



User-File-Model
101-7564-34553

1. INTRODUCTION

This is an automatically generated report for the stochastic projection model with identifier 101-7564-34553. It contains details of the model structure and the data to which it is fitted. Where technical terms are used, there is usually a link to the part of our website which documents the term and what it means.

The software used to fit this model and generate this report is the [Projections Toolkit](#). This is a service specifically tailored to the needs of analysts wanting to create stochastic mortality projections. The most common application is understanding the risks of mortality trends for pension schemes or annuity portfolios.

For analysing a portfolio's mortality-experience data, we recommend our companion service, the [Longevity survival-modelling system](#), which can determine sub-groups of a portfolio with different life expectancies.

2. LEGAL

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3. DATA

Table 1 summarises the data used in this report.

Table 1. Summary of data

Gender	M
Date range	1970–2005
Age range	55–90 years
Exposure type	initial

Figure 1 shows the number of deaths at each age.

Figure 1. Deaths by age.

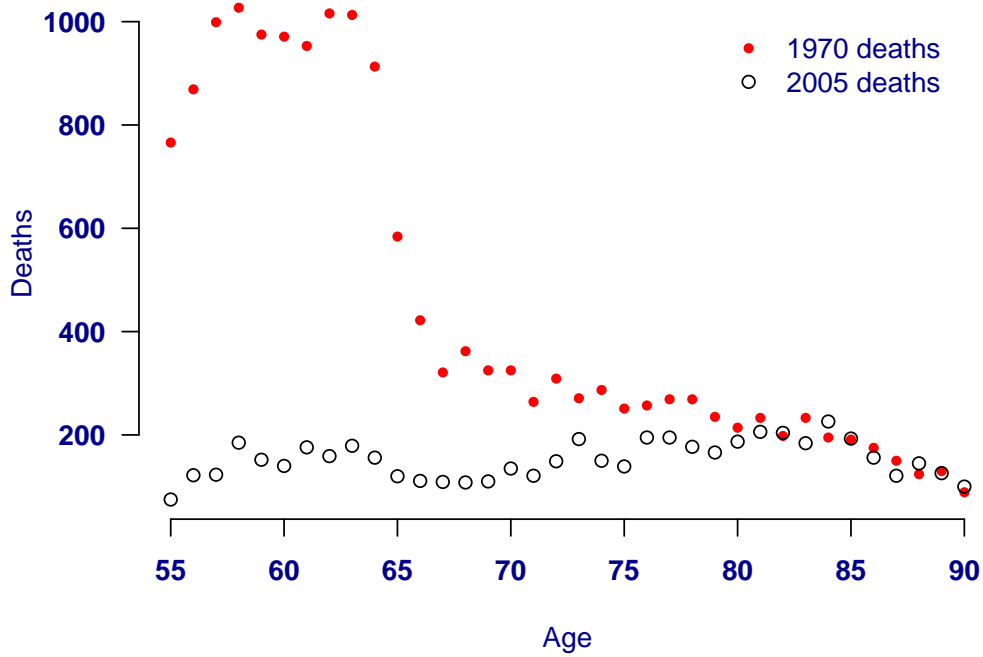


Figure 2 shows the mid-year population estimates by age.

The uploaded data were specified as initial exposure, i.e. data for a binomial-style model based on q_x . The exposures have been converted to make them suitable for a Poisson-style model based on μ_x .

Figure 2. Exposures by age.

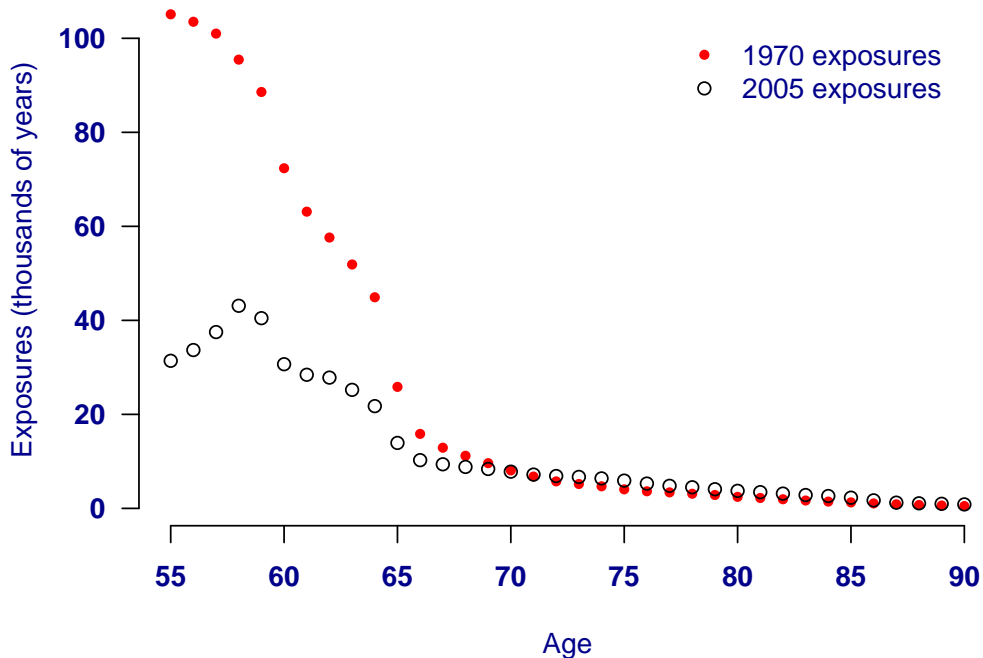


Figure 3 shows the raw mortality rates on a logarithmic scale. The roughness of the mortality surface will give an idea of how much smoothing is required, which may well vary by age.

Figure 3. Pseudo-3D plot of log(mortality rates) across range of ages and dates

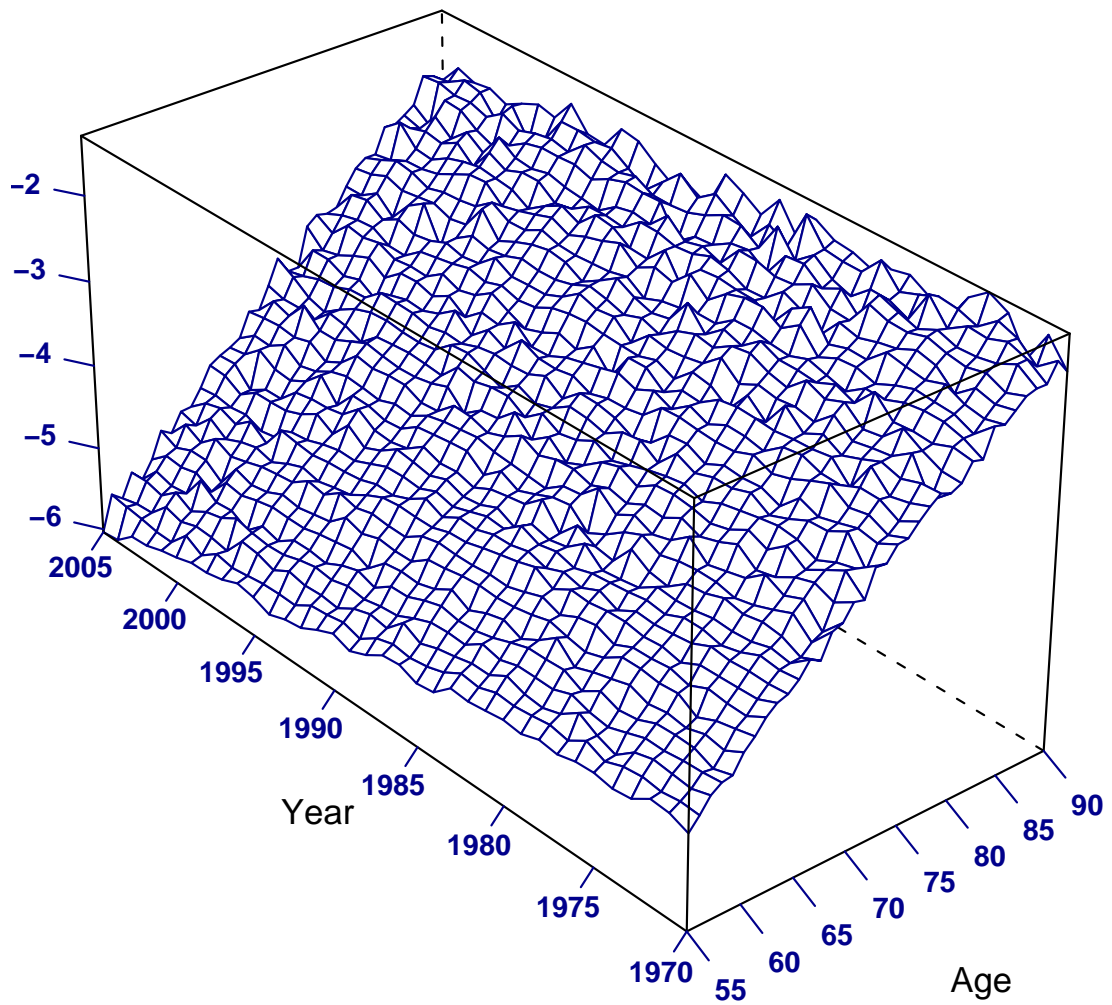
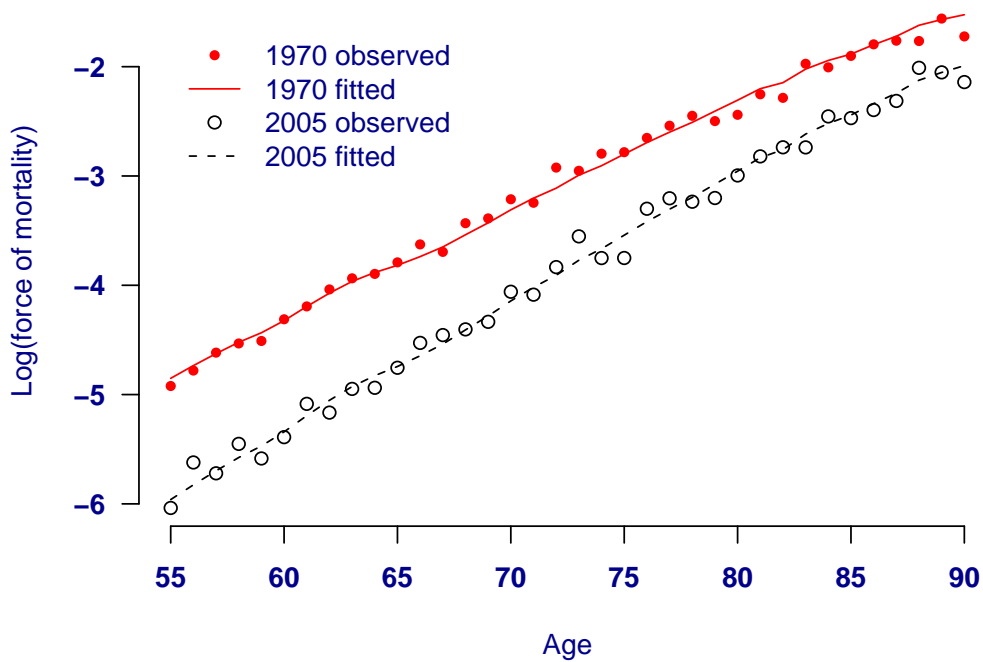


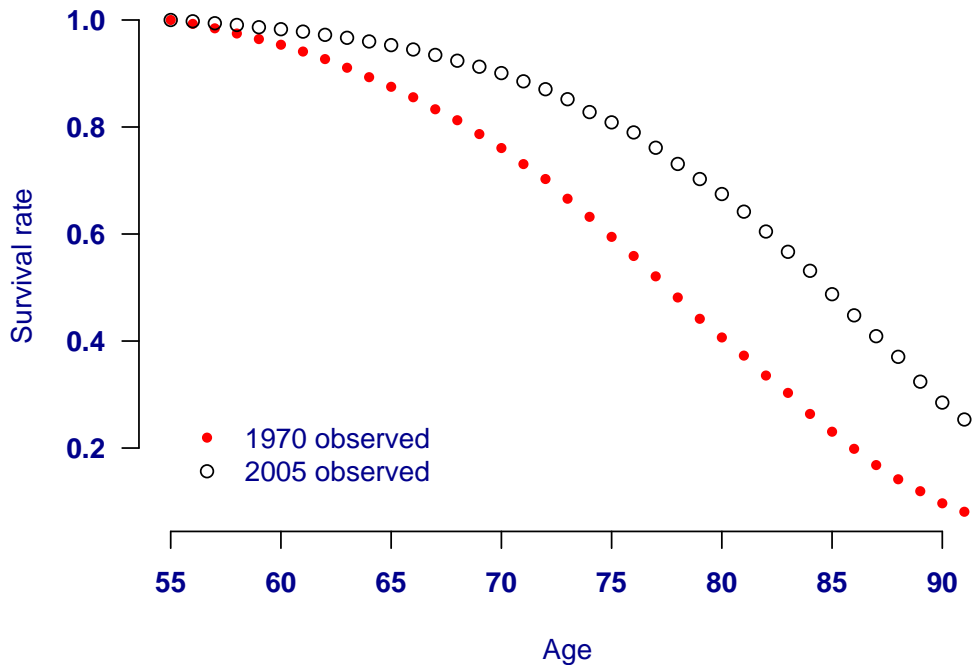
Figure 4 shows the raw and fitted mortality rates on a logarithmic scale.

Figure 4. Plot of period mortality rates



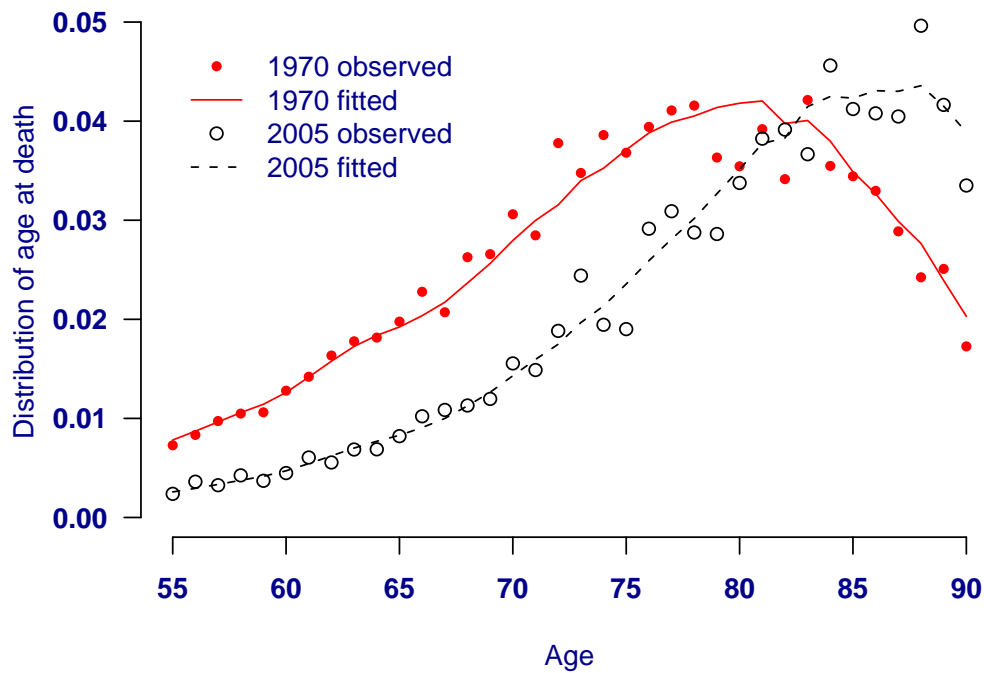
Another way to look at the data is to look at the survival curves for the rates applying at the start and end of the period. A survival curve shows the probability of surviving from age 55 to every subsequent age. A survival curve starts at 1 (or 100%) and reduces as the probability of surviving falls. Figure 5 shows the survival curve(s) for the mortality rates in this data set. The area under the survival curve is the expected time lived, so the area between the two survival curves is the difference in expected time lived.

Figure 5. Period survival curve(s) (${}_t p_{55}$)



Another way to look at the data is to look at the distribution of age at death, known to actuaries as the *curve of deaths*. If lifetime is viewed as a random variable, T , then the distribution of age at death is the truncated density function of T (${}_t p_{55} \mu_{55+t}$). Figure 6 shows curve(s) of deaths using the observed crude hazard rates. Note that the total area under the full curve will be 1, but if the upper age limit is low then you will only see the truncated density.

Figure 6. Distribution of age at death.



4. MODEL

Table 2 summarises the model used in this report.

Table 2. Summary of model

Model class	Lee-Carter
Model sub-class	smooth beta with ARIMA projection
Reference	Delwarde, Denuit and Eilers (2007)
BIC	1741.08
Optimization target	BIC

The basic model is that the number of deaths, D , has a Poisson distribution with mean $E \times \mu$, where E is the corresponding central exposure (waiting time) and μ is the force of mortality (mortality hazard). The model for μ follows Lee and Carter (1992), where mortality is assumed to have the structure:

$$\log \mu_{x,y} = \alpha_x + \beta_x \kappa_y$$

where α_x is the effect for age x and where β_x is the age-related response to the period effect κ_y in year y . The fitted values for α_x , β_x and κ_y are shown in Figure 7. The β_x are smoothed with penalised splines.

The Poisson assumption is modified by means of an over-dispersion parameter, ψ^2 , which is used to allow for more variation in the data than is allowed by the Poisson distribution. This extra-Poisson variation could arise from period effects (harsh winters or influenza epidemics, for example), or duplicate policies, or lack of homogeneity in the underlying lives. For more information on the over-dispersion parameter, see Djeundje and Currie (2011) or [our online journal](#).

Flexible localised smoothing is done by means of penalised splines. The technique is described in detail in Eilers and Marx (1996), and the details of the splines and penalty are tabulated below:

Spline degree	3
Spline spacing by age	5 years
Spline spacing in time	5 years
Penalty order	2
Smoothing constant for age, λ^{age}	9999998.341
Over-dispersion parameter, ψ^2	1.69

Projections are carried out by means of an ARIMA(3,1,3) time series applied to κ_y . The variance of the noise process, σ^2 , is 0.000142245528075258.

Parameter	Estimate	Std. err.
ar1	0.1	0.321
ar2	0.986	0.204
ar3	-0.097	0.264
ma1	-0.518	0.278
ma2	-0.851	0.309
ma3	0.627	0.213

The model parameters (or basis functions) are displayed in Figure 7. The convention is that discrete values are plotted with a single point, whereas values which are by design smooth or continuous are displayed as a solid line.

Figure 7. Plot of model parameters (or basis functions).

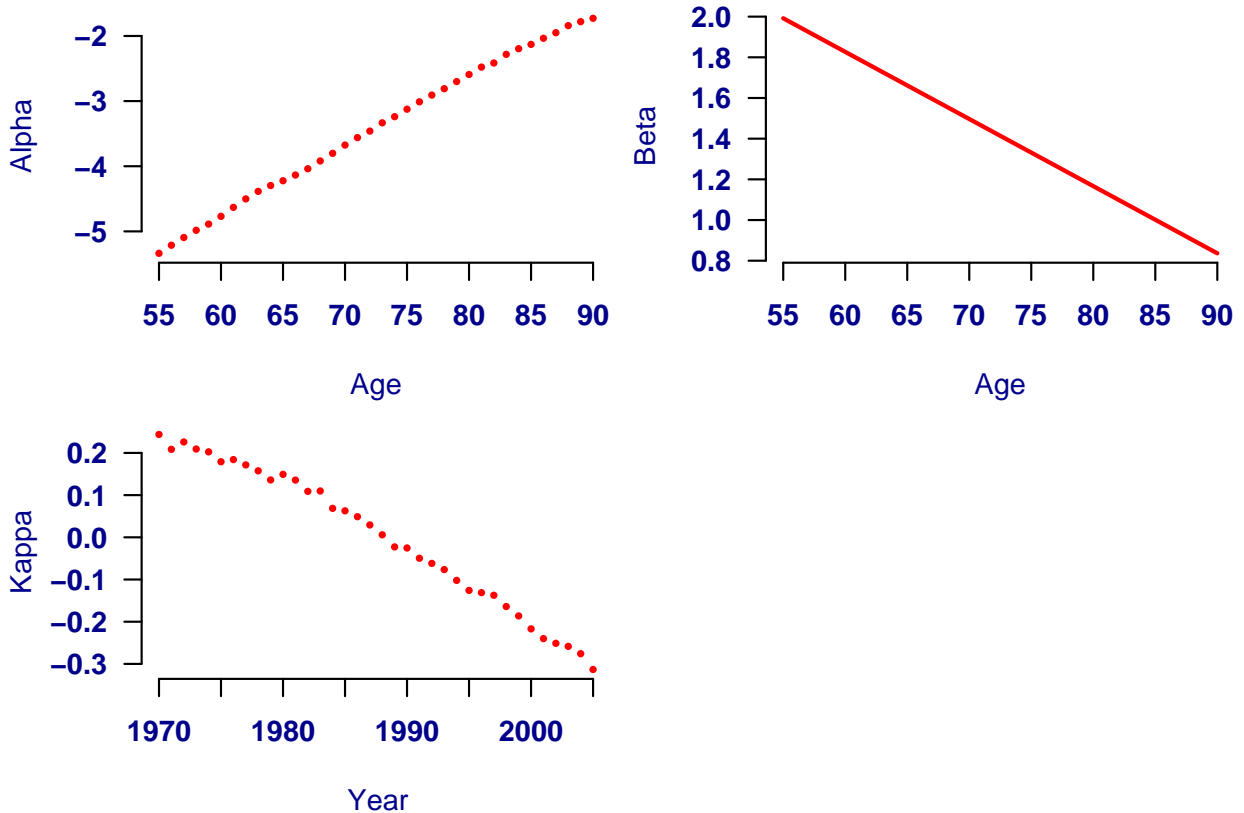


Figure 7.1 Plot of time-series alternatives with $d = 1$.

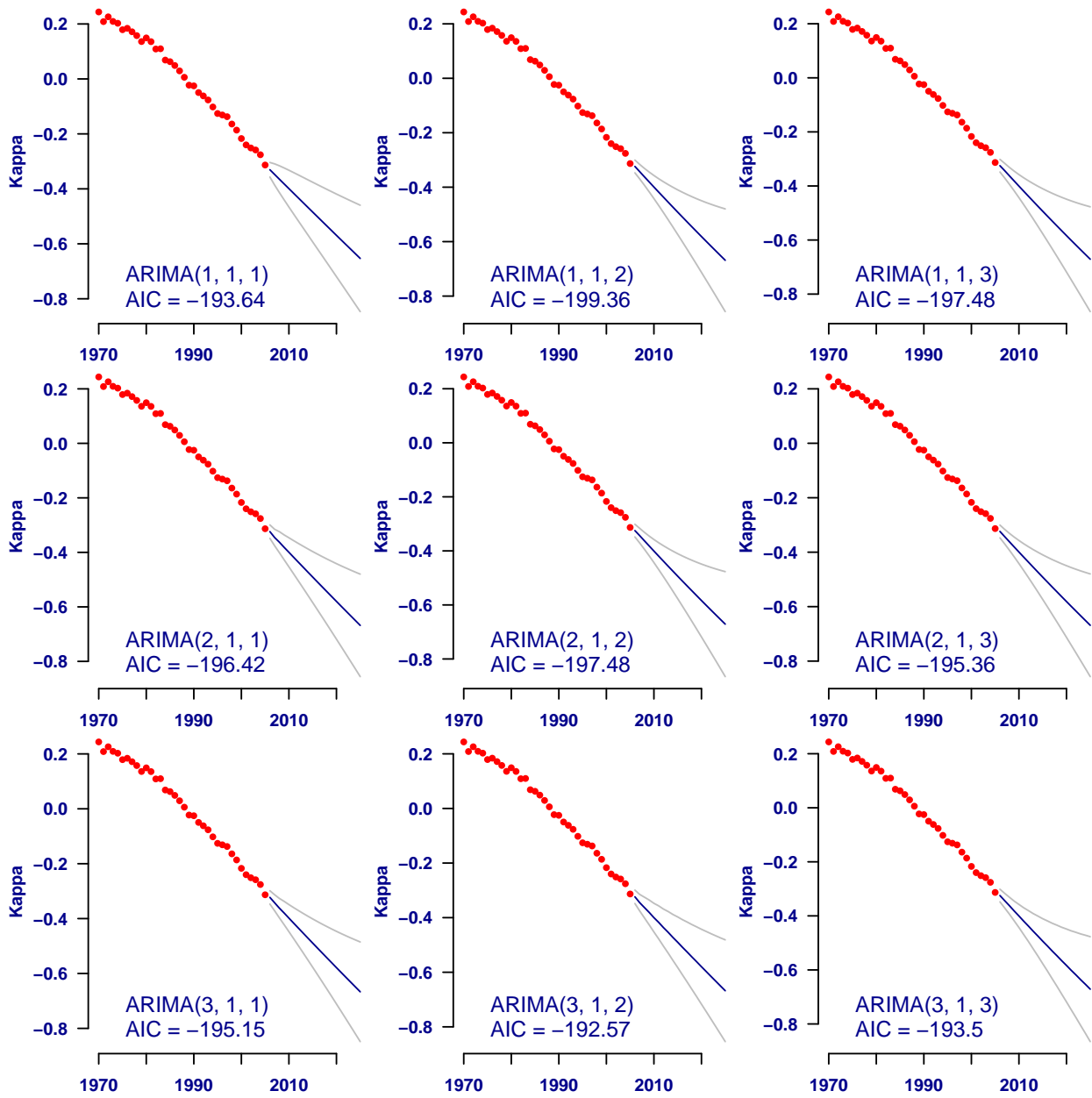
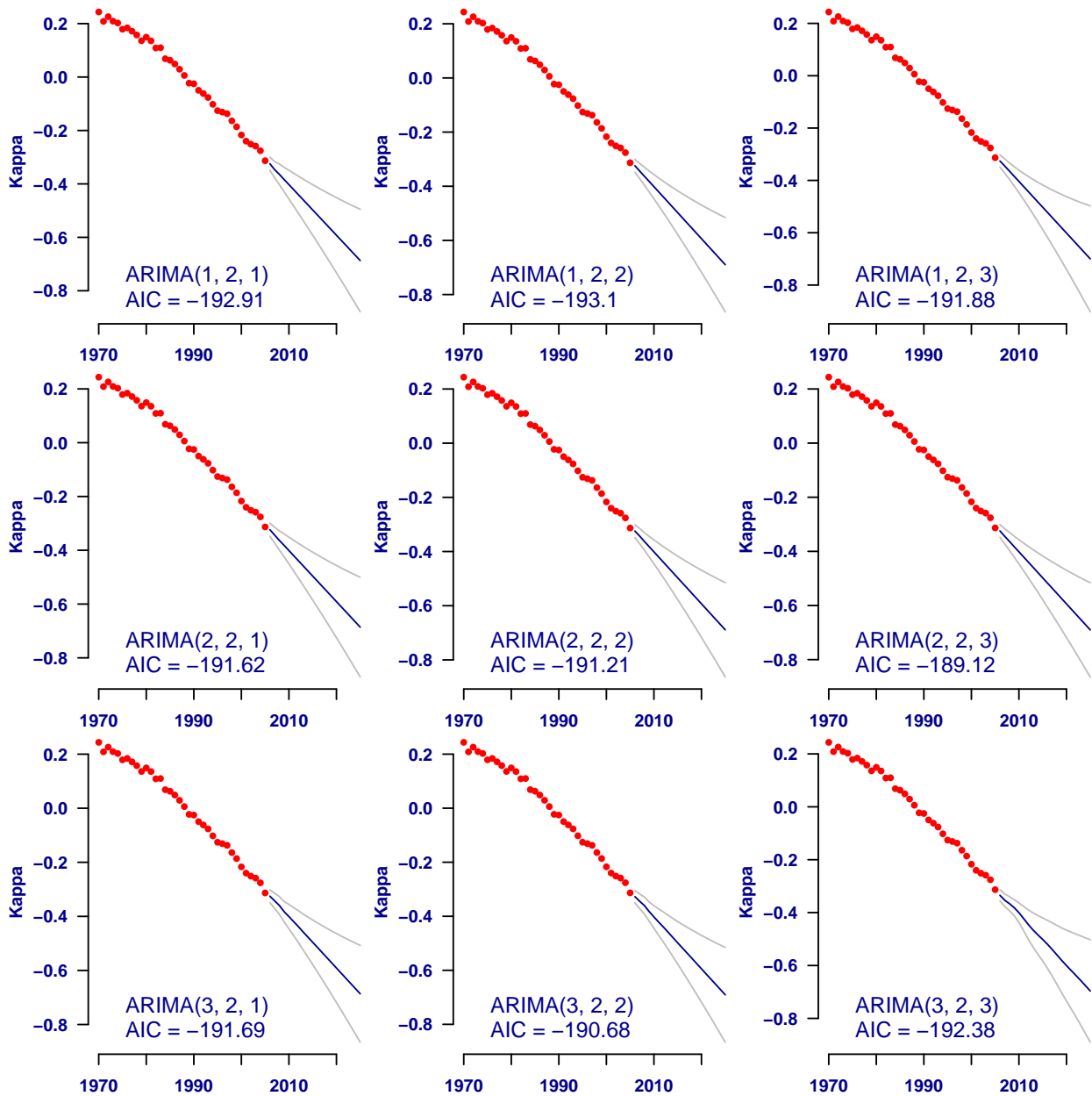


Figure 7.2 Plot of time-series alternatives with $d = 2$.



5. RESIDUALS

A key assumption of the model is that residuals are normally distributed, i.e. they are random variates drawn from the $N(0, 1)$ distribution. Figure 8 shows a density plot of the residuals for this model, together with a $N(0, 1)$ density. If the actual residuals form a squat-like version of the $N(0, 1)$ density with fatter tails outside the $N(0, 1)$ line, this indicates that there is more variability than the assumed $N(0, 1)$ assumption. For this model 4.01% of residuals lie outside ± 1.96 , against an expected proportion of 5% if the residuals were truly $N(0, 1)$.

Following McCullagh and Nelder (1989), we have used *deviance residuals*, which are similar to traditional Pearson residuals when the number of deaths is large but which are preferable when the number of deaths is small or if there are zero counts. The definition of a deviance residual, r , is:

$$r = \text{sign}(D - E\mu) \sqrt{2 \left(D \log \frac{D}{E\mu} - (D - E\mu) \right)}$$

where D is the number of observed deaths, E is the central exposed-to-risk and μ is the force of mortality.

The Poisson assumption here is modified by means of the over-dispersion parameter, ψ^2 , which was described in the

Model section. We have therefore modified the deviance residuals and used r^ψ , defined as:

$$r^\psi = \frac{r}{\psi}$$

Note that we do not perform any formal tests of fit, as these will largely be a waste of time — with such a large data set, no model will give an acceptable fit as there is simply too much power in the data. However, just because a model has poor residuals will not automatically mean the model is inappropriate for forecasting. You will have to use judgement in selecting a model.

Figure 8. Plot of residual density compared with $N(0, 1)$.

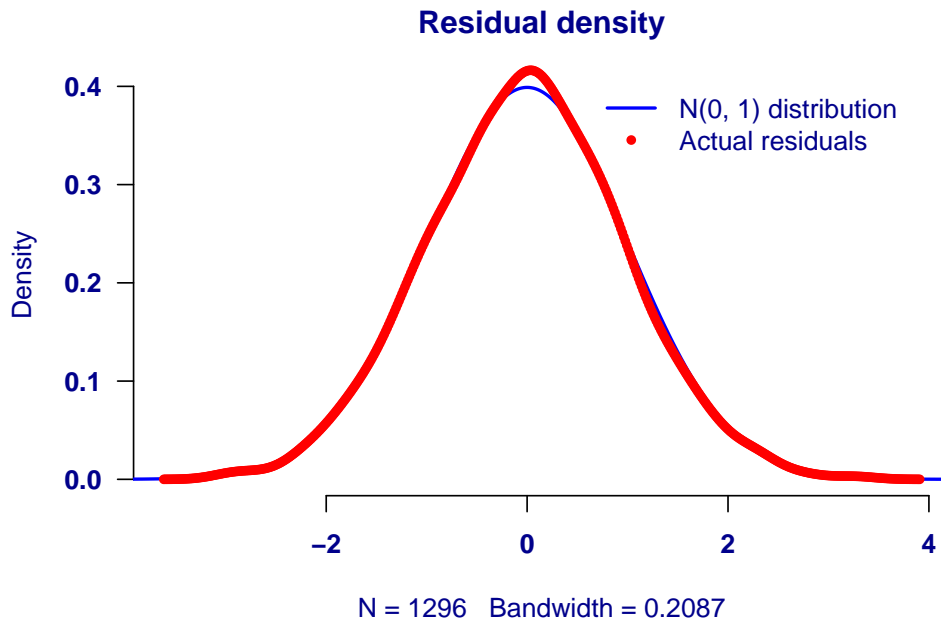


Figure 9 shows the plot of residuals by age. Ideally these residuals should show a random scatter consistent with $N(0, 1)$ variates, and there should be no strong patterns by age.

Figure 9. Plot of residuals by age.

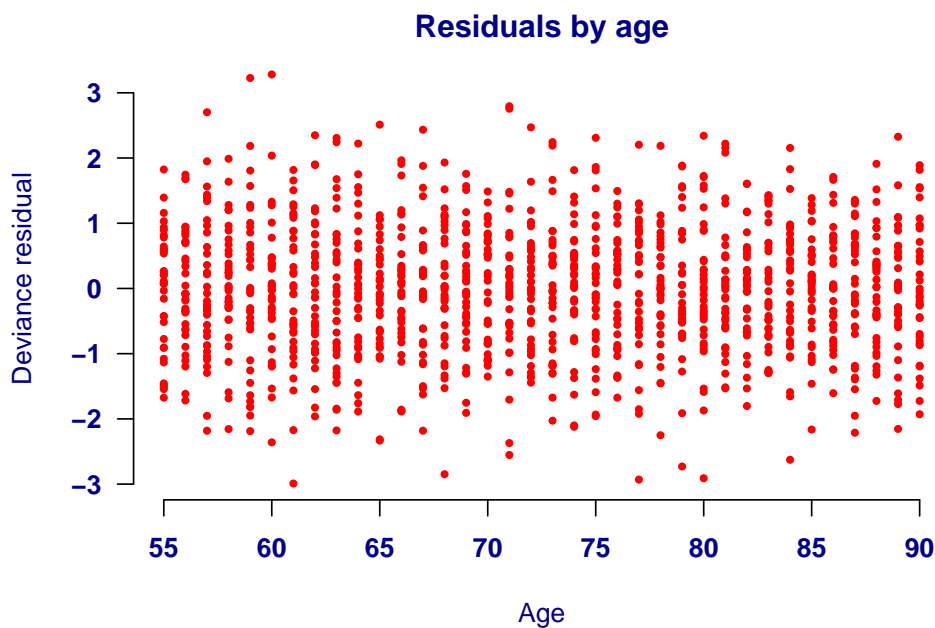


Figure 10 shows the plot of residuals by year. Ideally these residuals should show a random scatter consistent with $N(0, 1)$ variates, and there should be no strong patterns by year.

Figure 10. Plot of residuals by year.

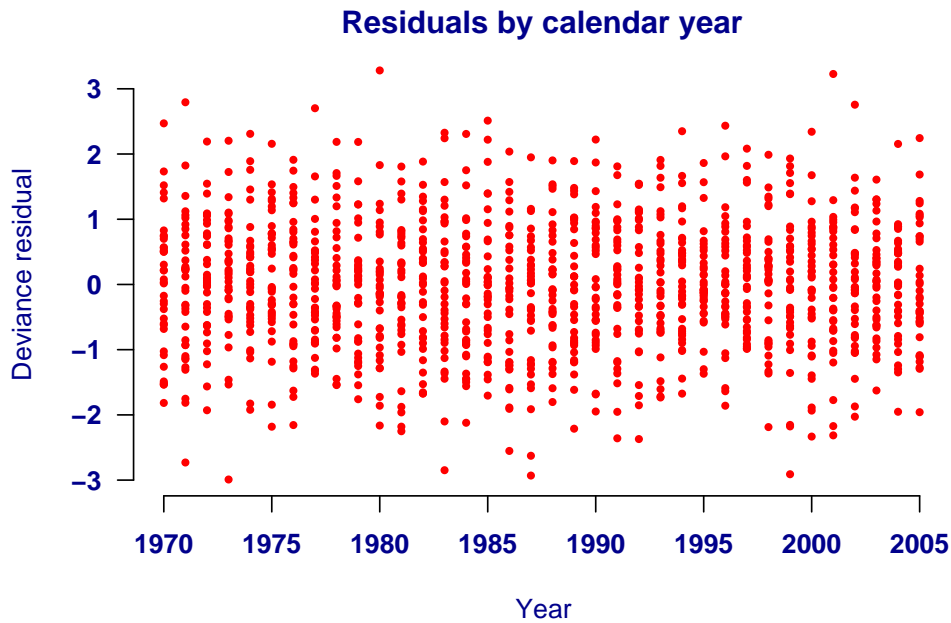
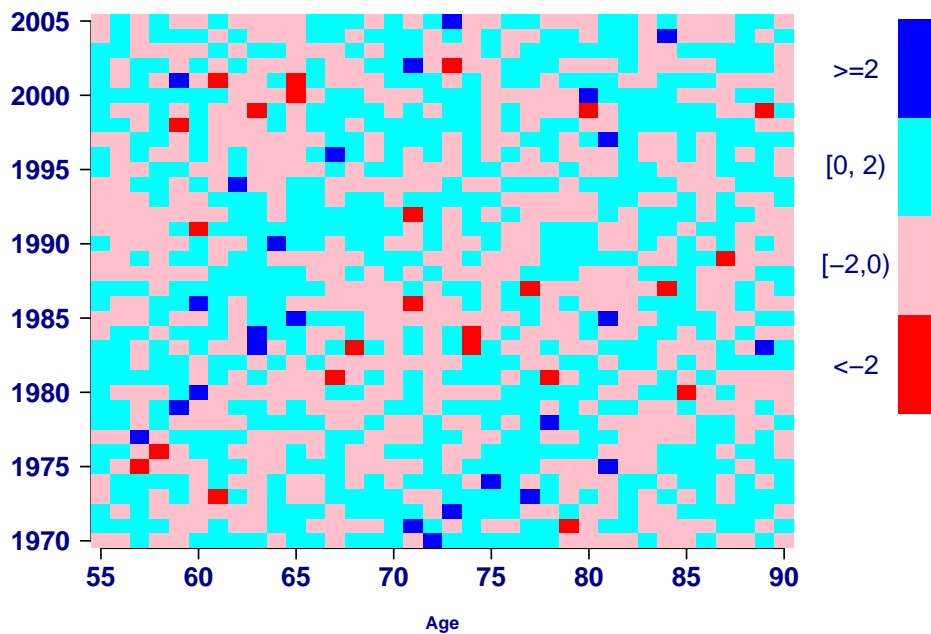


Figure 11 shows a two-dimensional plot of deviance residuals by age and year. There should be no large blocks of colour or patterns if the model is good.

Figure 11. Plot of residuals.



6. ILLUSTRATIVE VALUES AT KEY AGES

Table 3 shows the values for the time lived to age 91 at key ages. Table 4 shows the values for annuity to age 91 at key ages. All figures in Tables 3 and 4 are approximate continuous values for the calendar year stated, i.e. period calculations. Figures are given at the start and end of the data period, plus the end of the forecast period.

Table 3. Values for the time lived to age 91 at key ages (period mortality)

Age	1970	2005	2025
60	18.12	23.17	25.61
65	14.49	18.83	21
70	11.18	14.74	16.58
75	8.36	11.04	12.46

Table 4. Values for the annuity to age 91 at key ages, discounting at 5% p.a (period mortality).

Age	1970	2005	2025
60	11.27	13.3	14.19
65	9.72	11.76	12.7
70	8.06	10.01	10.96
75	6.45	8.15	9

Tables 3 and 4 are calculated using period mortality rates.

Tables 5 and 6 are equivalent to Tables 3 and 4, but use generational (or cohort) mortality.

Table 5. Values for the time lived to age 91 at key ages (cohort mortality)

Age	1970	2005	2025
60	19.55		
65	15.39		
70	11.7		
75	8.63	11.45	

Table 6. Values for the annuity to age 91 at key ages, discounting at 5% p.a. (cohort mortality)

Age	1970	2005	2025
60	11.77		
65	10.09		
70	8.32		
75	6.62	8.37	

Table 7 shows average annual improvement rates, i.e. the solution of c in $\mu_{x,y2} = (1 - c)^n \mu_{x,y1}$.

Table 7. Average annual improvements

Age	1970–2005	2005–2025	1970–2025
60	2.87%	3.22%	2.99%
65	2.61%	2.93%	2.73%
70	2.35%	2.64%	2.46%
75	2.1%	2.36%	2.19%

7. PROJECTIONS

Figure 12 shows the projected mortality rates compared to the fitted rates at the start and end of the data period. Figure 13 shows the projected period survival curve compared to the fitted survival curve(s).

Figure 12. Mortality rates.

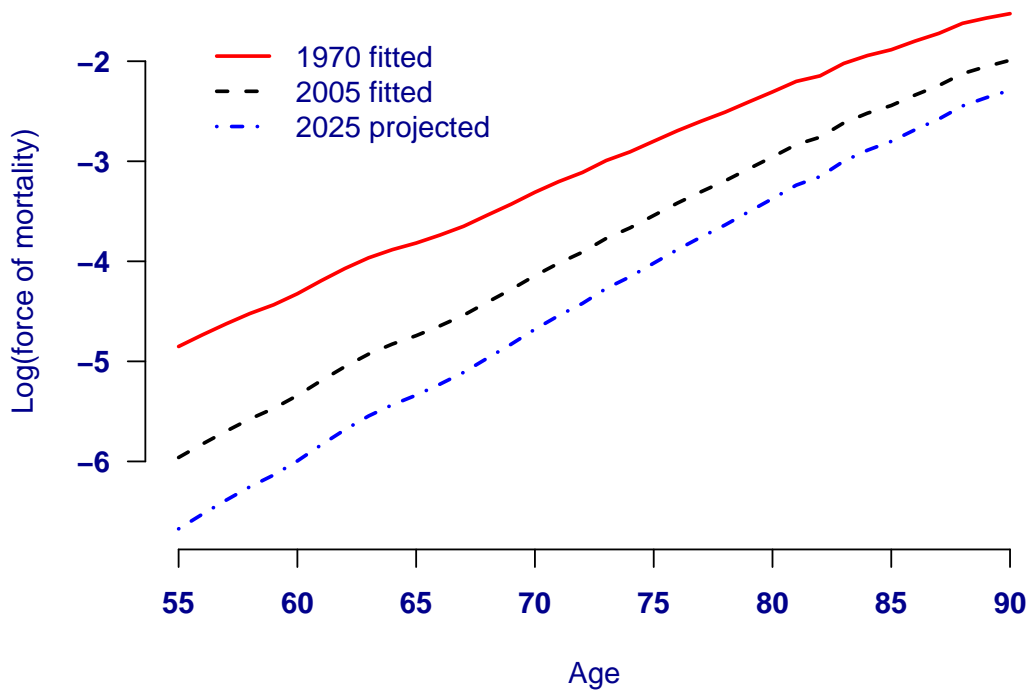


Figure 13. Period survival curve(s).

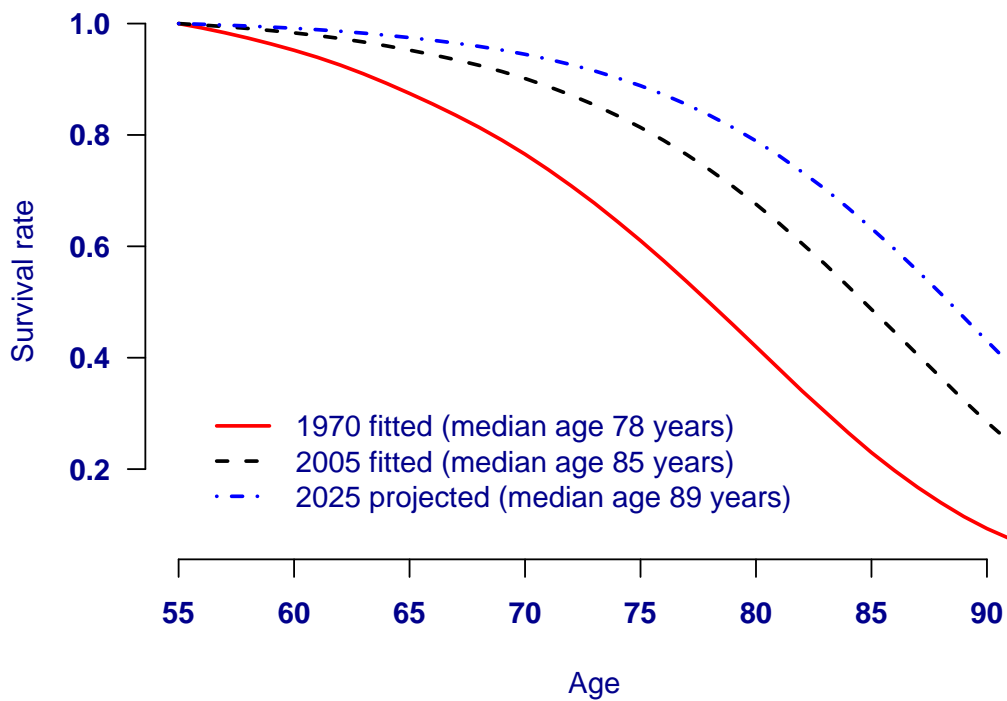


Figure 14. Period distribution(s) of age at death.

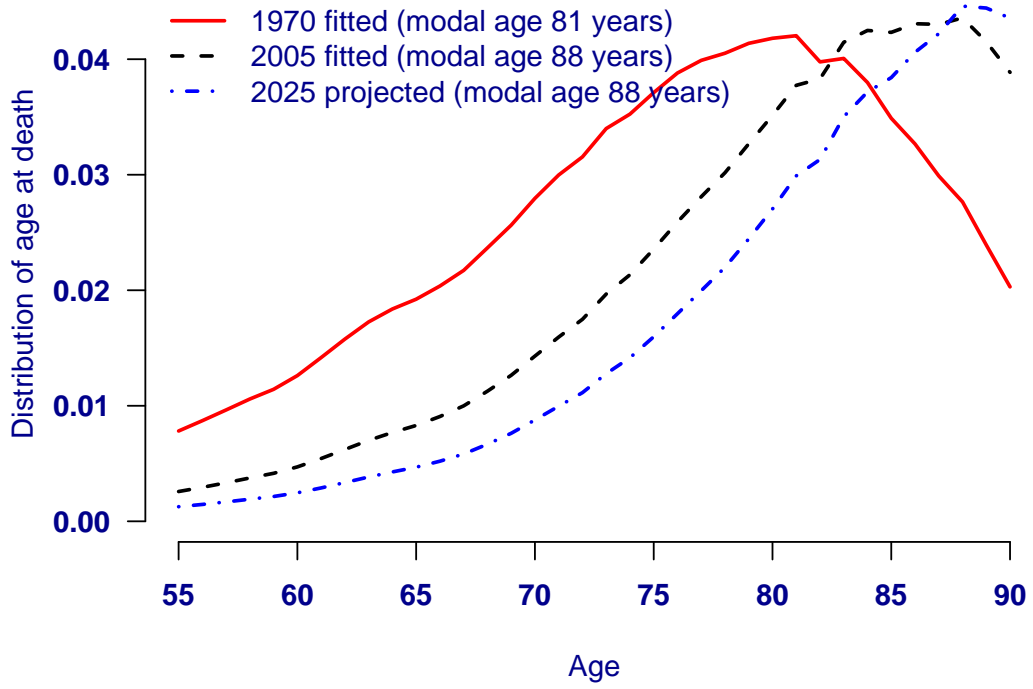


Figure 15. Mortality rates and projections at selected key ages

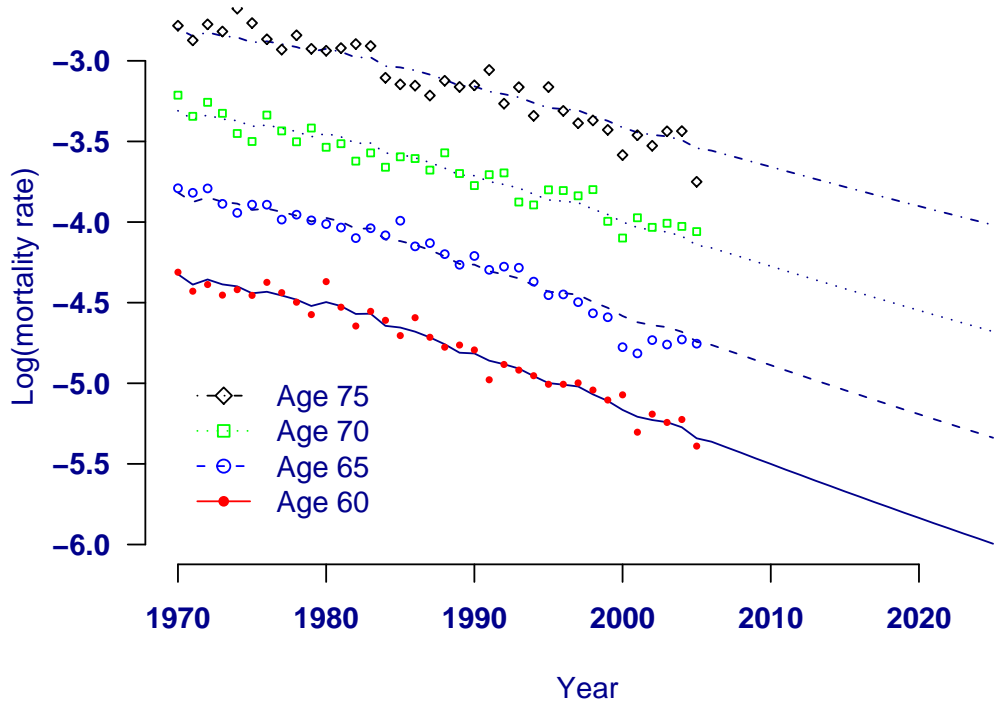


Figure 16. Mortality rates, projections and confidence intervals at selected key ages

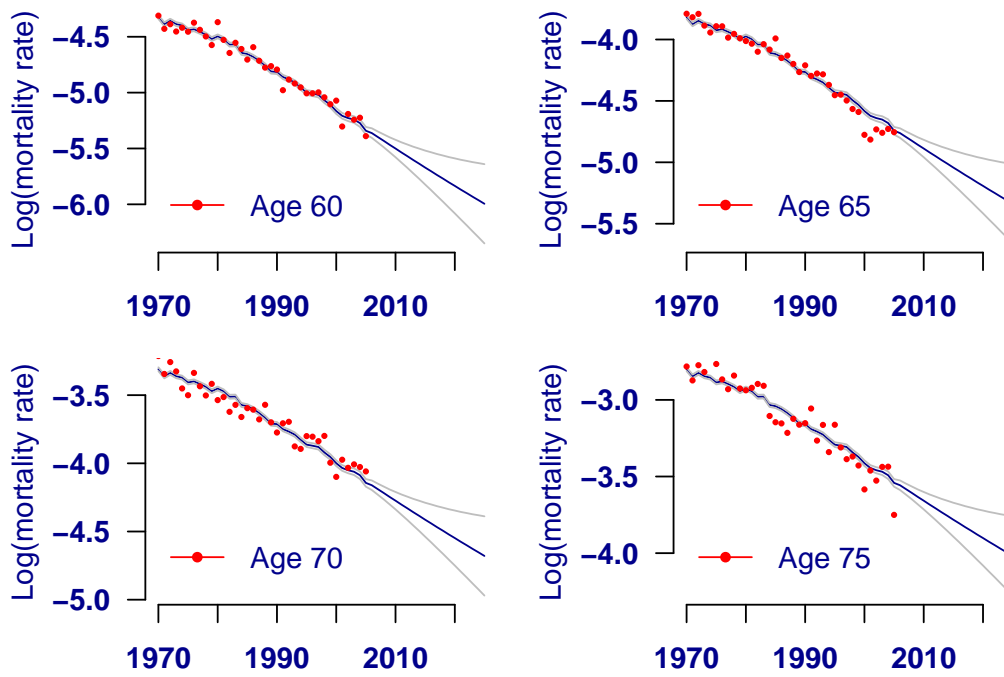
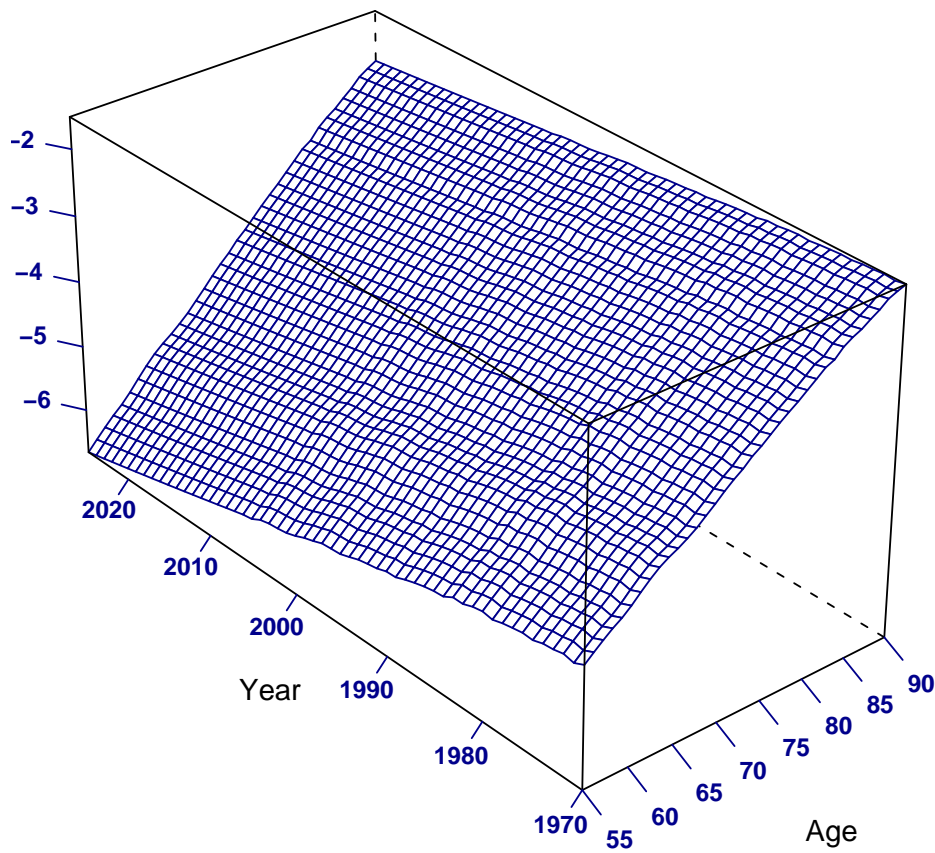


Figure 17. Pseudo-3D plot of fitted and projected mortality rates on a logarithmic scale



8. OTHER PROJECTION METHODOLOGIES AND OPTIONS

Longevity provides a suite of stochastic projection methodologies. We prefer such methods because they come with an explicit statement of uncertainty in the form of confidence intervals.

If you have sufficient portfolio experience data, you can build a model based directly on that. Such a model can also include time trends and cohort effects for projections. Using our [survival-modelling package](#) it is possible to model mortality trends within your own portfolio data and use these models as the basis for projections and portfolio run-off simulations for ICAs and Solvency II.

Our [online journal](#) contains further commentary on mortality projections at www.longevity.co.uk

9. TAS-D

This section is designed to help users who have to adhere to the standard for data used in actuarial work, [TAS-D](#), laid down by the [Board of Actuarial Standards](#) in the United Kingdom. It may be useful for users in other territories. The data used in generating this report might form part of actuarial work to which the stipulations of TAS-D will apply. The information that follows is to assist in the creation of such TAS-D-compliant material.

Data definitions (TAS-D C.5.3)

The data used are the counts of deaths in a given calendar year at the given age last birthday.

Validation (TAS-D C.5.6)

The data used in this report were validated when uploaded. The checks carried out include:

- Counts of deaths were non-negative.
- Population figures were non-negative.
- Deaths counts and population figures had matching ranges for age and calendar year.

If a file fails these data checks, then a model cannot be run.

10. TAS-R

This section is designed to help users who have to adhere to the standard for actuarial reports, [TAS-R](#), laid down by the [Board of Actuarial Standards](#) in the United Kingdom. It may be useful for users in other territories. Material within this report might form part of a component report within the definitions of TAS-R, and may therefore be used to assist in the creation of an aggregate report to which the stipulations of TAS-R will apply. The information that follows is to assist in the creation of such TAS-R compliant material.

10.1 Relevance (TAS-R C.3)

The purpose of this report is to describe the data and model used for a stochastic projections of future mortality rates. Such projections are used when valuing pension-scheme or annuity liabilities. Stochastic projection models can be used to produce a variety of alternative scenarios, each with a likelihood of occurring. These scenarios can be used to investigate the solvency position of a portfolio.

10.2 Data (TAS-R C.4.1)

The data used in fitting the model in this report is summarised in the section entitled “Data”. The data covers a range of ages and dates, and this may only be a sub-section of the available ages and dates. The data have undergone basic validation checks as follows:

- Counts of deaths and exposures are non-negative.
- Age ranges and date ranges for deaths and exposures match correctly.

10.3 Assumptions (TAS-R C.4.4)

The model fitted in this report is a statistical one for the number of deaths at each age, x , in a given year, y , $D_{x,y}$. This number is assumed to be a random variable following a [Poisson distribution](#):

$$D_{x,y} \sim \text{Poisson} \left(E_{x+\frac{1}{2},y+\frac{1}{2}}^c \times \mu_{x+\frac{1}{2},y+\frac{1}{2}} \right)$$

where $E_{x+\frac{1}{2},y+\frac{1}{2}}^c$ is the central exposed-to-risk, i.e. the mid-year population estimate in year y of lives at risk of dying aged x last birthday. $\mu_{x+\frac{1}{2},y+\frac{1}{2}}$ is the corresponding force of mortality (hazard rate). The model assumes a specific structure for $\mu_{x,y}$ which is documented in the section entitled “Model”.

10.4 Rationale (TAS-R C.4.6)

The model is fitted using the method of [maximum likelihood](#) (or maximum penalized likelihood in the case of models using smoothing splines). This is a statistical estimation procedure to find the best-fitting parameters for the model.

10.5 Uncertainty (TAS-R C.5.2)

As the output from a statistical model, the projections produced hereby are accompanied by standard errors. These are used to produce confidence intervals around the projections in Figure 16.

10.6 Probabilities (TAS-R C.5.13)

This report contains the probability of surviving from the earliest age in the data set to each of the subsequent ages. Such probabilities are called a [survival curve](#), examples of which are in Figures 5 and 13.

10.7 Projections (TAS-R C.5.20)

This report contains projections of possible future mortality trends. Figures 12–15 show three mortality curves: one each at the start and end of the data period, plus one projected curve at the end of the projection period. Figures 15 and 16 show actual observed mortality rates plus projected trend values. Figure 17 shows the fitted mortality surface in the data region together with projected surface.

There are two important additional risks which are beyond the scope of this report:

1. Model risk, i.e. the fact that it is impossible to know if the model used in this report is the correct one. Since the confidence intervals in Figure 16 are largely determined by the choice of model, they are critically vulnerable to the risk that the choice of model is incorrect.

2. Basis risk, i.e. the fact that the data used to parameterise the model used in this report may not be identical to any portfolio valued using these projections.

11. TAS-M

This section is designed to help users who have to adhere to the standard for actuarial models, [TAS-M](#), laid down by the [Board of Actuarial Standards](#) in the United Kingdom. It may be useful for users in other territories.

Checks (TAS-M C.3.5)

The data used in parameterising the projection model in this report were subjected to numerous checks:

- Counts of deaths were non-negative.
- Population figures were non-negative.
- Deaths counts and population figures had matching ranges for age and calendar year.

If a data file fails any one of these checks, then models cannot be run.

Neutral measures (TAS-M C.3.10)

The parameter estimates in the section entitled “Model” and elsewhere are maximum-likelihood estimates.

Parsimony (TAS-M C.3.14)

The number of parameters in a model is under the user’s control, but the BIC in Table 2 can be used to compare models and ensure that each parameter is necessary in the model by balancing fit against model complexity. Penalised-spline smoothing can be added to most models to reduce the effective number of parameters in a model.

Reproducibility (TAS-M C.3.17)

The model fitted in this report is a regression-type model, i.e. a fit of a model to the data. As long as the data and model selection are unchanged, the fit will be reproducible.

Assumptions (TAS-M C.4.18)

The key assumption underlying the model described in this report is that the death counts for each combination of age and calendar year are independent from each other. Where this assumption is deemed not to be valid, for many models the user can introduce an over-dispersion parameter to counter the resulting effects of the failure of independence. If an over-dispersion parameter is fitted, the section entitled “Model” documents how it is used and what value has been used.

12. REFERENCES AND FURTHER READING

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We publish regular online articles on [mortality projections](#) on our website at www.longevity.co.uk.

13. ENVIRONMENT

This section documents the configuration parameters applying when the model in this report was fitted. These options are under user control in the Projections Toolkit under the menu option Settings→Configuration→Projections.

Table 10. Summary of environmental variables

Dataset Gender	M
Generate Residuals File	NO
Mortality Rate Interpolation	linear
Projection Autoregressive Order P	3
Projection Convergence Tolerance	0.0000001
Projection Data End Age	90
Projection Data End Year	2005
Projection Data Start Age	55
Projection Data Start Year	1970
Projection Degree Of Differencing D	1
Projection Fitting Step Size	0.001
Projection Fitting Target	df
Projection Interest Rate	5
Projection Iteration Limit	50
Projection Key Ages	60,65,70,75
Projection Model	LC DDE TS
Projection Optimisation Method	L-BFGS-B
Projection Order Of Moving Average Q	3
Projection Overdispersion	0
Projection Override Optimal ARIMA	YES
Projection Random Number Generator	Knuth
Projection Spline Age Knot Positioning	first
Projection Spline Age Knot Spacing	5
Projection Spline Degree	3
Projection Spline Order	2
Projection Spline Year Knot Positioning	last
Projection Spline Year Knot Spacing	5
Projection Target DF 1D Models	2
Projection Target DF 2D Models	100
Projection Year	2025
Residual Weighting Threshold	0
Run Date	2011-10-14
Tracking	101 7564 34553
Version	v0.672 2011-08-25
Statistics module return code	0
Executable termination code	0