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A VaR approach to mis-estimation risk

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Overview

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1. Mis-estimation risk
2. Portfolio features
3. Parameter estimation
4. Preconditions
5. Run-off mis-estimation
6. Value-at-risk mis-estimation
7. Comparisons
8. Conclusions

1 Mis-estimation risk

1 Longevity risk

> "the PRA considers that longevity risk includes at least two sub-risks [...] namely, base mis-estimation risk and future improvement risk"
>
> Woods [2016]

1 Mis-estimation risk

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"[. . .] the risk that the base mortality estimate is incorrect (i.e. the mortality estimate based on actual experience in the portfolio)" Burgess et al. [2010]

"How wrong could our base mortality assumptions be, or: what if our historical experience did not reflect the underlying mortality?" Armstrong [2013]

1 Measuring mis-estimation

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"Mis-estimation risk lends itself to statistical analysis if there is sufficient accurate data" Armstrong [2013]

"The impact of uncertainty should always be quantified financially" Makin [2008]

1 Mis-estimation risk

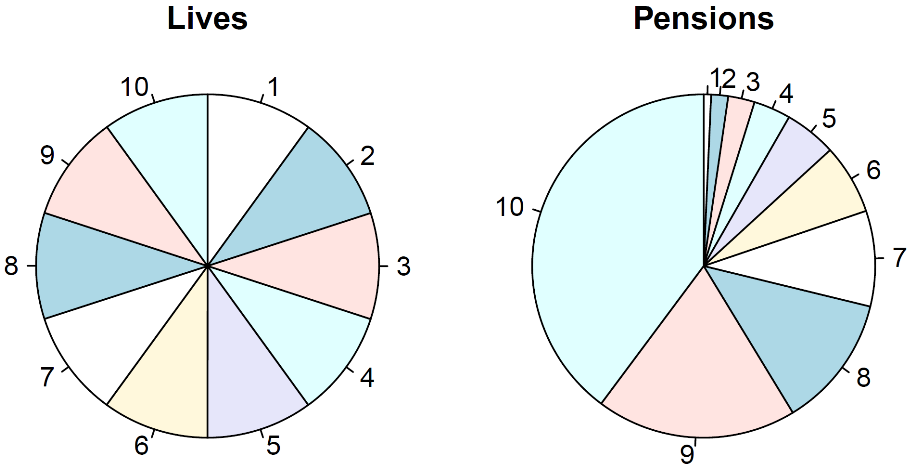
- Uncertainty over current mortality rates,
- Assessed using actual portfolio experience,
- Modelled statistically, and
- Measured financially.

2 Portfolio features

2 Portfolio features

- Medium-sized pension scheme: 15,698 lives.
- Pensioners only.
- 5,956 deaths in 2007–2012.

2 Liability concentration



Source: Data from Richards [2016, Appendix 1].

2 Estimation risk

- Top decile of pensioners receives 39.8% of pensions.
- Next two deciles receive further 31.4%.
- Liabilities highly concentrated.

2 Example model

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Consider a time-varying model for the mortality hazard:

$$\mu_{x,y} = \frac{e^{\epsilon} + e^{\alpha+\beta x+\delta(y-2000)}}{1 + e^{\alpha+\beta x+\delta(y-2000)}}$$

where x is exact age and y is calendar time[†].

[†] -2000 is an offset to keep parameters well scaled.

2 Example model

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Consider simple approach to gender and pension size for life i:

$$\begin{aligned}\alpha_i = \alpha_0 &+ \alpha_{\text{male}} z_{i,\text{male}} \\ &+ \alpha_{\text{decile 8 or 9}} z_{i,\text{decile 8 or 9}} \\ &+ \alpha_{\text{decile 10}} z_{i,\text{decile 10}}\end{aligned}$$

- α_j is the effect of risk factor j.
- $z_{i,j}$ is an indicator taking the value 1 if life i has risk factor j and zero otherwise.

2 Portfolio features

Parameter	Estimate	Std. Err	Lives
β	0.148	0.005	15,698
α_{male}	0.479	0.060	5,956
α_0	-14.731	0.491	15,698
ϵ	-5.420	0.154	15,698
$\alpha_{\text{decile 8 or 9}}$	**-0.180**	0.078	3,140
$\alpha_{\text{decile 10}}$	**-0.313**	0.108	1,567
δ	-0.046	0.016	15,698

Source: Parameter estimates from Richards [2016, Table 6].

2 Coefficient of variation

$$\text{Coefficient of variation} = \frac{\text{Standard error}}{|\text{Estimate}|}$$

Measures relative uncertainty over parameter estimate.

2 Coefficient of variation

Parameter	Coef. of variation
β	0.03
α_{male}	0.13
α_0	0.03
ϵ	0.03
$\alpha_{\text{decile 8 or 9}}$	**0.43**
$\alpha_{\text{decile 10}}$	**0.35**
δ	0.35

Source: own calculations from estimates in Richards [2016, Table 6].

2 Estimation risk

- Liabilities highly concentrated.
- Sub-groups with most liability have lowest mortality...

  ... *and* high relative uncertainty.

3 Parameter estimation

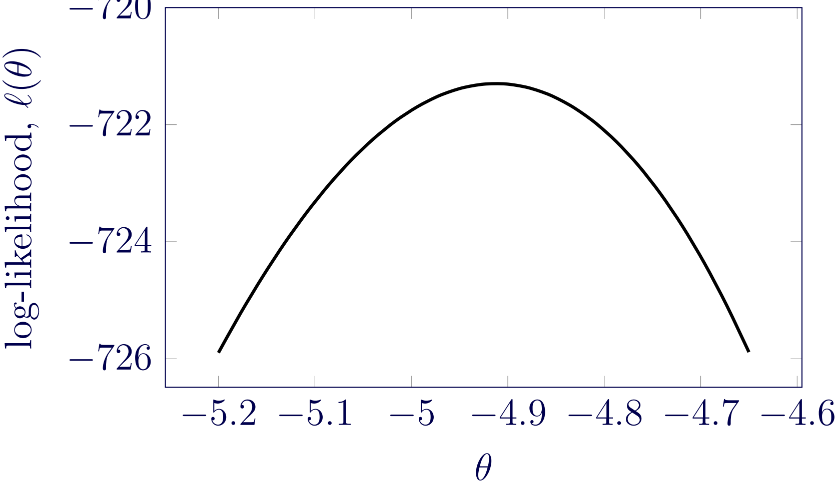
In a statistical model with m parameters:

- Consider a parameter vector, $\underline{\boldsymbol{\theta}} = \begin{pmatrix} \boldsymbol{\theta}_1 \\ \boldsymbol{\theta}_2 \\ \vdots \\ \boldsymbol{\theta}_m \end{pmatrix}$.
- We have an estimate, $\underline{\hat{\boldsymbol{\theta}}}$, which is uncertain.
- Uncertainty over $\underline{\hat{\boldsymbol{\theta}}}$ is estimation risk.

3 Assumption 1

$\ell(\underline{\boldsymbol{\theta}})$ is the log-likelihood function for a model.

3 $\ell(\theta)$ in one dimension



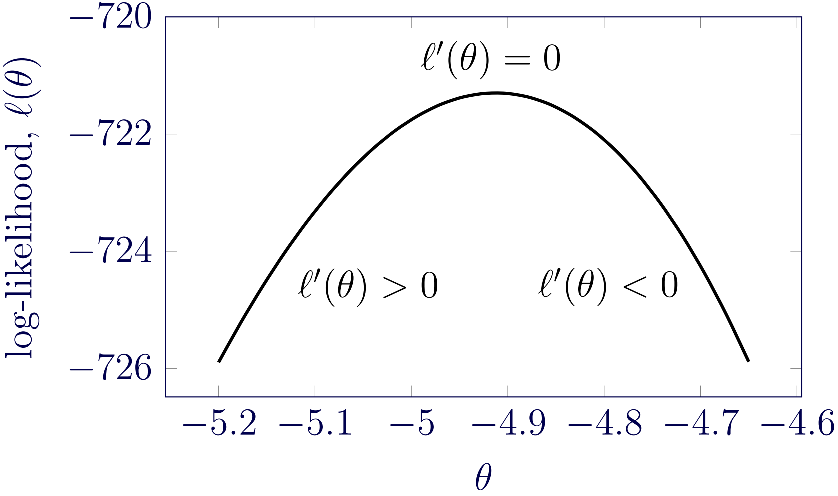
Source: Richards [2016, Figure 1].

3 Assumption 2

All first partial derivatives of $\ell(\underline{\boldsymbol{\theta}})$ exist, i.e.

$$\ell'(\underline{\boldsymbol{\theta}}) = \begin{pmatrix} \dfrac{\partial \ell(\underline{\boldsymbol{\theta}})}{\partial \theta_1} \\ \dfrac{\partial \ell(\underline{\boldsymbol{\theta}})}{\partial \theta_2} \\ \vdots \\ \dfrac{\partial \ell(\underline{\boldsymbol{\theta}})}{\partial \theta_m} \end{pmatrix}$$

3 $\ell(\theta)$ in one dimension



Source: Richards [2016, Figure 1].

3 Assumption 3

The Hessian matrix, $\boldsymbol{H}(\underline{\boldsymbol{\theta}})$, of all second partial and cross-partial derivatives of $\ell(\underline{\boldsymbol{\theta}})$ exists:

$$\boldsymbol{H}(\underline{\boldsymbol{\theta}}) = \begin{pmatrix} \frac{\partial^2 \ell(\underline{\boldsymbol{\theta}})}{\partial \theta_1^2} & \frac{\partial^2 \ell(\underline{\boldsymbol{\theta}})}{\partial \theta_1 \partial \theta_2} & \cdots & \frac{\partial^2 \ell(\underline{\boldsymbol{\theta}})}{\partial \theta_1 \partial \theta_m} \\ \frac{\partial^2 \ell(\underline{\boldsymbol{\theta}})}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 \ell(\underline{\boldsymbol{\theta}})}{\partial \theta_2^2} & \cdots & \frac{\partial^2 \ell(\underline{\boldsymbol{\theta}})}{\partial \theta_2 \partial \theta_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \ell(\underline{\boldsymbol{\theta}})}{\partial \theta_m \partial \theta_1} & \frac{\partial^2 \ell(\underline{\boldsymbol{\theta}})}{\partial \theta_m \partial \theta_2} & \cdots & \frac{\partial^2 \ell(\underline{\boldsymbol{\theta}})}{\partial \theta_m^2} \end{pmatrix}$$

3 Maximum likelihood

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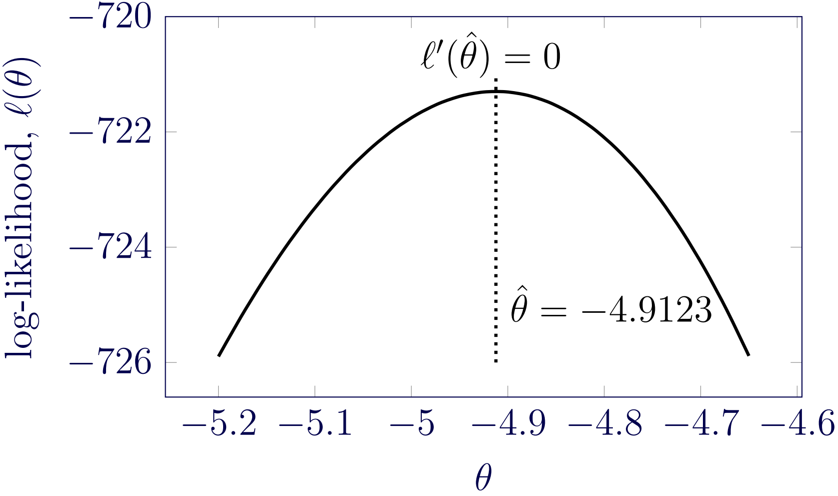
$\underline{\hat{\boldsymbol{\theta}}}$ is the maximum-likelihood estimate of $\underline{\boldsymbol{\theta}}$ if:

- $\ell'(\underline{\hat{\boldsymbol{\theta}}}) = 0$, and
- $\boldsymbol{H}(\underline{\hat{\boldsymbol{\theta}}})$ is negative semi-definite†.

† $\underline{\boldsymbol{x}}^{\mathrm{T}}\boldsymbol{H}(\underline{\hat{\boldsymbol{\theta}}})\underline{\boldsymbol{x}} \leq 0, \forall \underline{\boldsymbol{x}} \in \mathbb{R}^m$, where $\underline{\boldsymbol{x}}^{\mathrm{T}}$ denotes the transpose of $\underline{\boldsymbol{x}}$.

3 $\ell(\theta)$ in one dimension



Source: Richards [2016, Figure 1].

3 Maximum-likelihood theorem

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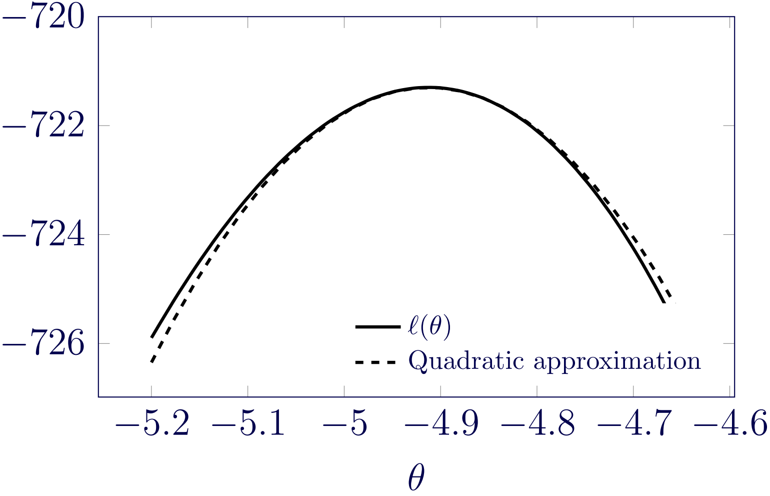
$\hat{\underline{\boldsymbol{\theta}}}$ has an approximate multivariate normal (MVN) distribution†:

- Mean vector $\hat{\underline{\boldsymbol{\theta}}}$, and
- Covariance matrix $\hat{\boldsymbol{\Sigma}} = -\boldsymbol{H}^{-1}(\hat{\underline{\boldsymbol{\theta}}})$.

† Cox and Hinkley [1996, Chapter 9].

3 $\ell(\theta)$ in one dimension



Source: Richards [2016, Figure 1].

3 Estimation risk

If $\underline{\hat{\boldsymbol{\theta}}}$ has a MVN distribution, all estimation risk is summarised in $\hat{\boldsymbol{\Sigma}}$:

- The leading diagonal has the variance of $\underline{\hat{\boldsymbol{\theta}}}$, and
- The off-diagonal entries have the covariances of $\underline{\hat{\boldsymbol{\theta}}}$.

3 (Mis-)Estimation risk

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Estimation risk

Statistical parameter uncertainty.

Mis-estimation risk

Financial impact of parameter uncertainty.

3 (Mis-)Estimation risk

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Estimation risk

Uncertainty over $\hat{\underline{\boldsymbol{\theta}}}$.

Mis-estimation risk

Uncertainty over $V(\hat{\underline{\boldsymbol{\theta}}})$, where V is the liability function depending on $\hat{\underline{\boldsymbol{\theta}}}$.

4 Preconditions

4 Preconditions

"What assumptions are you making, e.g. independence? Duplicate policies? Amounts vs lives?"
Armstrong [2013]

4 Preconditions

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- Deduplicate records†.
- Lives-based statistical model.
- Amounts effect on mortality handled as either:
  - Categorical factor, e.g. pension decile, or
  - Continuous covariate, e.g. using exact pension‡.

† See Macdonald et al. [2018, Section 2.5].

‡ See Richards [2020a].

4 Preconditions

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Mis-estimation capital underestimated if:

- Records not deduplicated, or
- Amounts effect on mortality ignored, or
- Time trend not included in model.

5 Run-off mis-estimation

5 Run-off mis-estimation

What is the uncertainty over $V(\hat{\underline{\boldsymbol{\theta}}})$ caused by the uncertainty over $\hat{\underline{\boldsymbol{\theta}}}$?

5 Run-off mis-estimation

Two approaches to uncertainty over $V(\underline{\hat{\boldsymbol{\theta}}})$:

- Delta method.
- Richards [2016].

5 Run-off 1 — delta method

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- $\underline{\hat{\boldsymbol{\theta}}}$ is a random variable.
- $V(\underline{\hat{\boldsymbol{\theta}}})$ is a function of a random variable.
- $\text{Var}[V(\underline{\hat{\boldsymbol{\theta}}})] \approx \underline{\boldsymbol{a}}^T \hat{\boldsymbol{\Sigma}} \underline{\boldsymbol{a}}$, where $\underline{\boldsymbol{a}} = \dfrac{\partial V(\underline{\hat{\boldsymbol{\theta}}})}{\partial \underline{\hat{\boldsymbol{\theta}}}}$.

5 Run-off 2 — Richards [2016]

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- Need set, $\mathcal{S}$, of liability valuations subject to parameter risk.
- Can then calculate percentiles of $\mathcal{S}$.

- Best-estimate parameter vector $\underline{\hat{\boldsymbol{\theta}}}$.
- $\hat{\boldsymbol{\Sigma}}$ is estimated covariance matrix for $\underline{\hat{\boldsymbol{\theta}}}$.
- Alternative parameter vector, $\underline{\boldsymbol{\theta}}'$, can be sampled from $\text{MVN}(\underline{\hat{\boldsymbol{\theta}}}, \hat{\boldsymbol{\Sigma}})$ using Monte Carlo simulation.
- Liability value $V(\underline{\boldsymbol{\theta}}')$ added to set $\mathcal{S}$.

5 Run-off 2 — Richards [2016]

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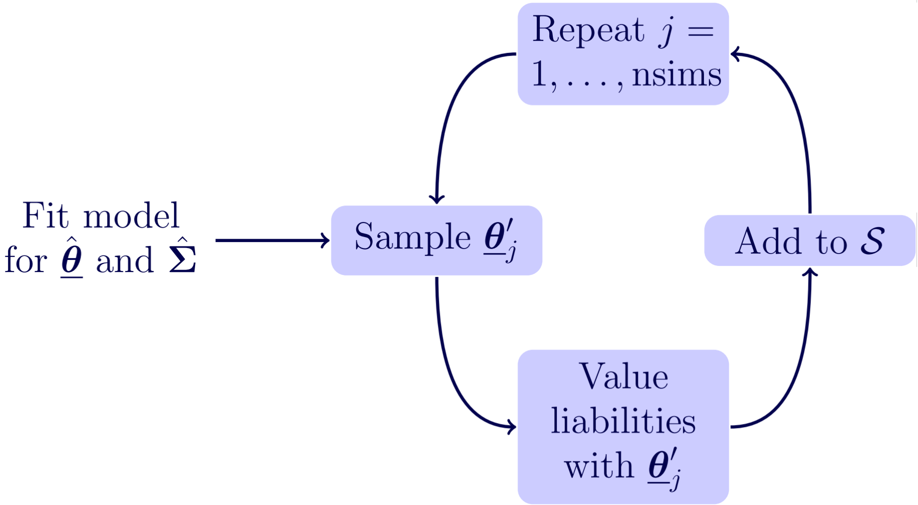
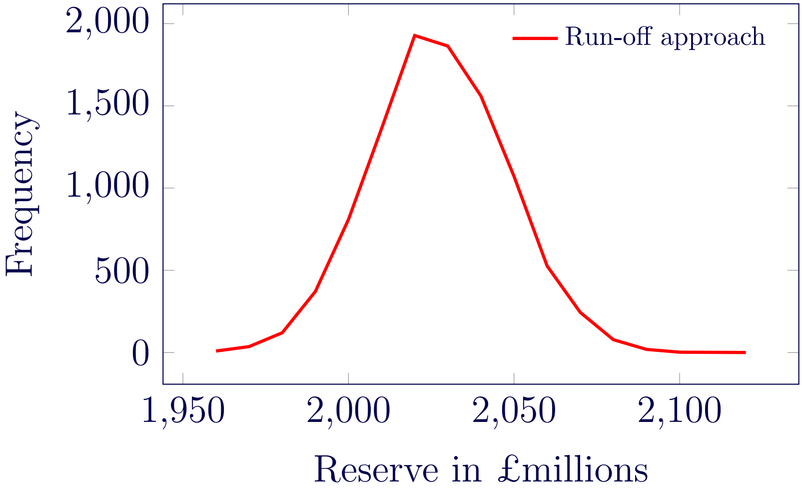


5 Second portfolio

- Larger pension scheme: 44,616 lives.
- Pensioners only.
- 10,663 deaths in 2001–2009.

5 Distribution of $\mathcal{S}$



Source: Richards [2020c, Figure 4].

What is the relative capital percentage, RCP, required to cover a proportion p of mis-estimation risk?

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$$RCP = \Phi^{-1}(p)\frac{\sqrt{\underline{\boldsymbol{a}}^T\hat{\boldsymbol{\Sigma}}\underline{\boldsymbol{a}}}}{V(\hat{\underline{\boldsymbol{\theta}}})} \times 100\%$$

where $\Phi^{-1}(p)$ is the inverse of the normal distribution function.

Note that $\frac{\sqrt{\underline{\boldsymbol{a}}^T\hat{\boldsymbol{\Sigma}}\underline{\boldsymbol{a}}}}{V(\hat{\underline{\boldsymbol{\theta}}})}$ is the coefficient of variation.

5 RCP under Richards [2016]

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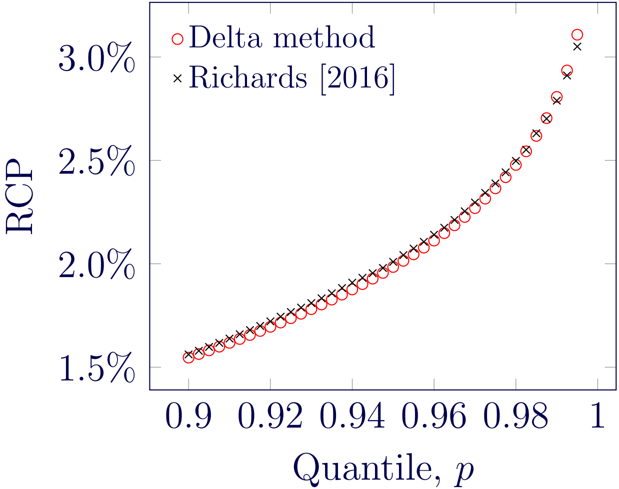
$$RCP = \left[\frac{Q_p(\mathcal{S})}{V(\underline{\hat{\boldsymbol{\theta}}})} - 1\right] \times 100\%$$

where $Q_p(\mathcal{S})$ is the p-quantile of $\mathcal{S}$.

5 Run-off mis-estimation capital



Source: Own calculations using model in Richards [2020c, Table 4].

5 Run-off mis-estimation

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- Both methods agree closely for this liability, even far into the upper tail.
- Delta method is quicker.
- Richards [2016] is better for skewed sets $\mathcal{S}$.

6 Value-at-risk mis-estimation

6 Run-off v. value-at-risk

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Run-off mis-estimation

What is the uncertainty over $V(\underline{\hat{\boldsymbol{\theta}}})$ caused by the uncertainty over $\underline{\hat{\boldsymbol{\theta}}}$?

VaR mis-estimation

What is the uncertainty over $V(\underline{\hat{\boldsymbol{\theta}}})$ caused by the re-estimation of $\underline{\hat{\boldsymbol{\theta}}}$ with n years additional data?

The VaR question is about recalibration risk [Cairns, 2013].

6 Run-off v. VaR approach

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The run-off approach:

- Has no "new experience" element...
- Has no recalibration element...
- Is not a plug-in fit for VaR applications like Solvency II.

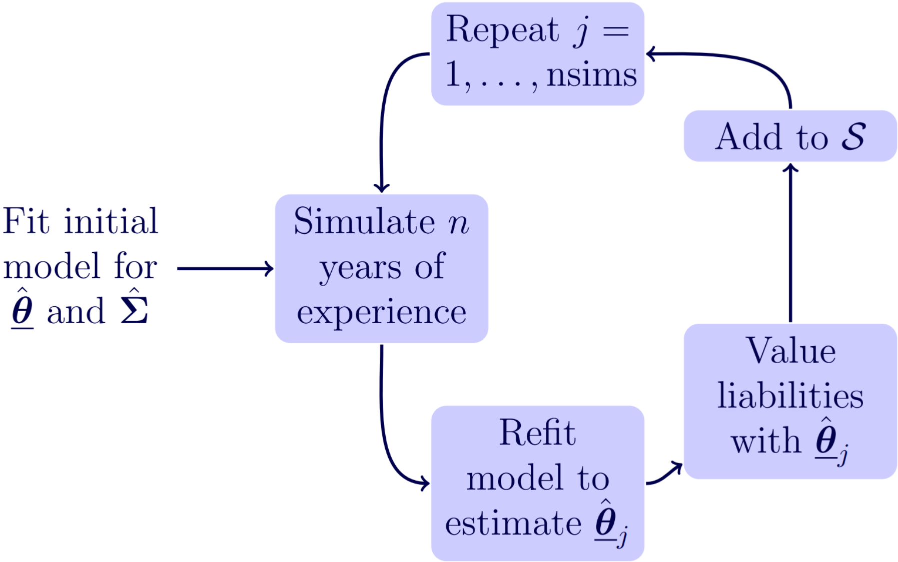
6 Value-at-risk mis-estimation

We make two changes to the algorithm of Richards [2016]:

1. Simulate n years of experience.
2. Refit model to re-estimate $\underline{\hat{\boldsymbol{\theta}}}_j$.

and proceed to generate set $\mathcal{S}$ as before.

6 Richards [2020c]



6 Richards [2020c]

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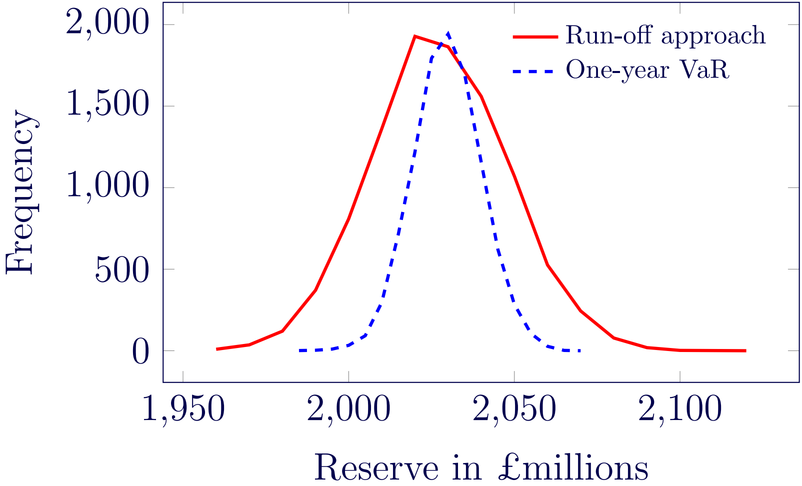
- $\underline{\hat{\boldsymbol{\theta}}}_j$ varies according to additional n years of experience data.
- True n-year VaR approach to mis-estimation.
- Calculate percentiles of $\mathcal{S}$ as before.

6 Distribution of $\mathcal{S}$



Source: Richards [2020c, Figures 4 and 6].

7 Comparisons

7 Model

- Survival model varying in age and time ($\mu_{x,y}$).
- Risk factors in model†:
  - Age
  - Gender
  - Normal v. early retirement
  - First life v. surviving spouse
  - Pension size
  - Time

† Source: Richards [2020c, Table 3].

7 Comparisons

- Interested in 99.5% VaR capital requirement.
- VaR capital $= \left(\frac{99.5\% \text{ percentile of } \mathcal{S}}{\text{Mean of } \mathcal{S}} - 1 \right) \times 100\%$.
- 10,000 samples or simulations for each scenario.

7 Richards [2020c]

Two options when simulating n years of experience:

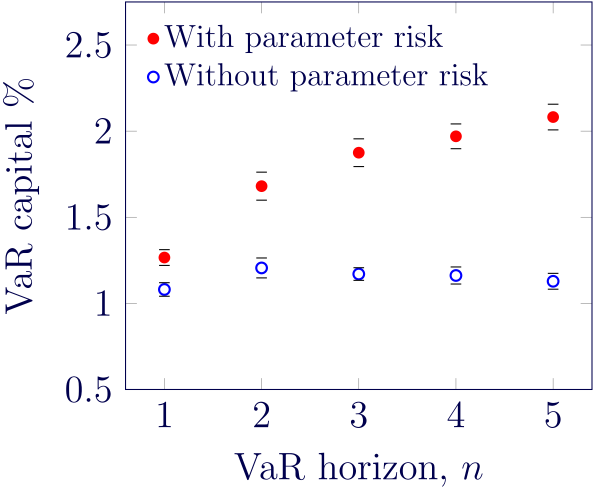
Without parameter risk

Use $\hat{\underline{\boldsymbol{\theta}}}$ each time.

With parameter risk

Sample from MVN$(\hat{\underline{\boldsymbol{\theta}}}, \hat{\boldsymbol{\Sigma}})$.

7 VaR capital by horizon



Source: Richards [2020c, Figure 5]. 99.5% VaR capital requirement from 10,000 simulations.

7 VaR capital by horizon

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For mis-estimation VaR capital:

- Including parameter risk increases capital.
- However, most of 1-year capital *not* due to parameter risk.
- Only half of 5-year capital due to parameter risk.

7 Solvency II correlations

- Solvency ($n = 1$) different from ORSA ($n = 3, 4, 5$).
- How correlated is VaR mis-estimation capital with idiosyncratic risk?

7 Correlations

Correlations between the mis-estimation VaR reserves at $y_1 = 2010$ and three measures of the simulated pseudo-experience underlying the recalibration.

VaR horizon	Time lived	New deaths	Payments made
1	80.2%	-90.1%	76.5%
2	80.8%	-88.9%	80.0%
3	81.1%	-87.7%	82.5%
4	81.7%	-86.9%	84.8%
5	81.7%	-86.2%	86.4%

7 Correlations

- VaR mis-estimation capital strongly correlated with idiosyncratic risk.
- Must be reflected in aggregation matrix for Solvency II.

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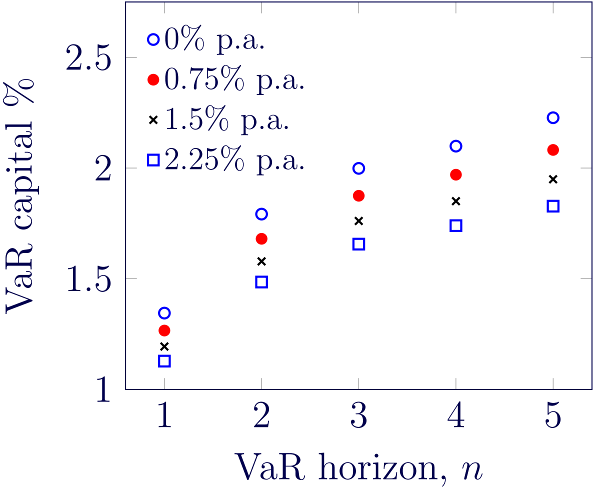

Source: Richards [2020c, Figure 7]. 99.5% VaR capital requirement from 10,000 simulations.

7 Role of discount rate

Lower discount rates mean higher mis-estimation VaR capital requirements.

7 Role of mortality law

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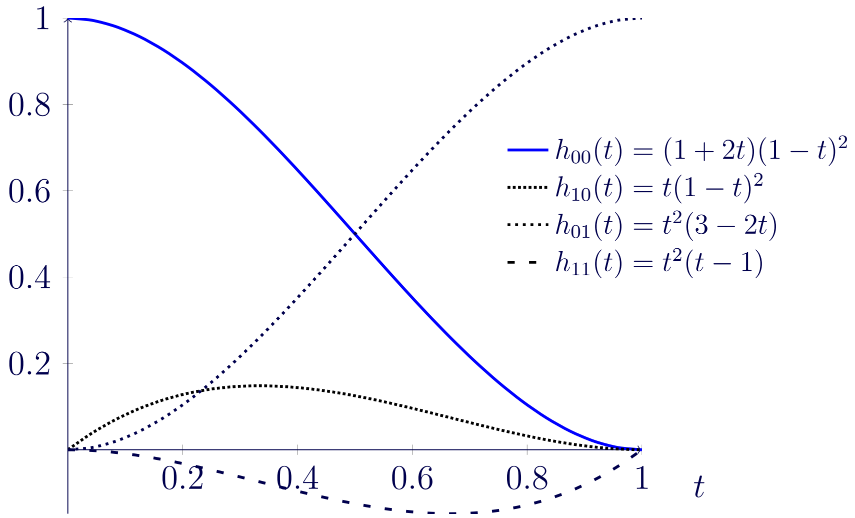
Various options for post-retirement mortality:

- Gompertz [1825], $\mu_x = e^{\alpha+\beta x}$.
- Perks [1932], $\mu_x = \dfrac{e^{\alpha+\beta x}}{1+e^{\alpha+\beta x}}$.
- Beard [1959], $\mu_x = \dfrac{e^{\alpha+\beta x}}{1+e^{\alpha+\rho+\beta x}}$.
- Makeham-Perks, $\mu_x = \dfrac{e^{\epsilon}+e^{\alpha+\beta x}}{1+e^{\alpha+\beta x}}$.
- Hermite-spline model of Richards [2020b].

7 A Hermite-spline model

- Designed for post-retirement mortality.
- Automatically allows for convergence with age.
- Uses Hermite splines for smoothness and flexibility.

7 A basis of Hermite splines



Source: Richards [2020b].

7 Hermite-spline model

- Let x_0 and x_1 be the minimum and maximum ages.
- Define $t = \frac{(x - x_0)}{(x_1 - x_0)}$, so $t \in [0, 1]$.
- $\log \mu_x = \alpha h_{00}(t) + \omega h_{01}(t) + m_0 h_{10}(t) + m_1 h_{11}(t)$

for parameters α, ω, m_0 and m_1 estimated from data.

Source: Richards [2020b].

7 Hermite-spline model

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- Mortality differentials in α reduce with age due to path of h_{00}.

  $\rightarrow$ don't need age interactions as in other models.

  $\rightarrow$ requires fewer parameters.

7 Questions

1. Which models fit best?
2. Do fewer parameters mean less mis-estimation risk?

7 Measures of model fit

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By lives

Use information criterion like AIC [Akaike, 1987].

By amounts

Use pension-weighted resampling approach like bootstrapping [Macdonald et al., 2018, Section 6.7].

7 What is the best fit?

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By lives

Lowest AIC.

By amounts

Bootstrap percentage closest to 100%.

7 Model fits

Sorted by descending AIC:

Mortality law	Parameters	AIC	Bootstrap
Gompertz	13	79,643	98.9%
Perks	13	79,638	99.3%
Beard	14	79,626	99.0%
Makeham-Perks	14	79,625	99.0%
Hermite	10	79,623	99.2%

Source: Data from Richards [2020c, Table 4].

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- Gompertz model has worst fit in terms of both AIC and explanation of amounts variation.
- Hermite model has:
    - Fewest parameters,
    - Best fit in terms of AIC, and
    - Second-best fit in terms of amounts.

7 Role of mortality law

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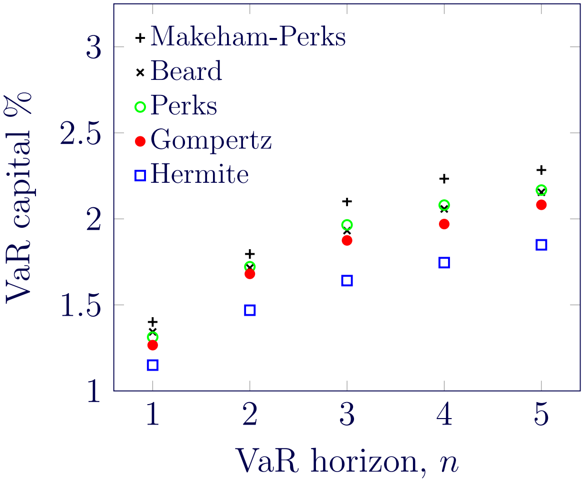

Source: Richards [2020c, Figure 10]. 99.5% VaR capital requirement from 10,000 simulations.

7 Role of mortality law

- Models with fewer parameters have lower mis-estimation risk.
- Overly simple laws (Gompertz) don't fit well enough.

8 Conclusions

8 Conclusions — I

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- Mis-estimation risk stems from having finite experience data.
- Quantification must be:
  - statistical to account for correlations, and
  - financial to account for concentration risk.

Two alternative views of mis-estimation:

Run-off

For pricing block transactions like reinsurance treaties, bulk annuities or longevity swaps.

Value-at-risk

For Solvency II reporting and ORSA.

8 Conclusions — III

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For one-year VaR mis-estimation capital:

- Parameter risk is a surprisingly minor driver.
- Instead, recalibration risk dominates.
- Strong correlation with idiosyncratic risk.

8 Acknowledgements

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