Longevitas client webinar

Estimation and valuation in Longevitas

Stephen J. Richards

26th January 2023

Overview

1. Motivation
2. Estimation and likelihood
3. Liability valuation
4. Valuation uncertainty
5. Conclusions

1 Motivation

1 Motivation

Q1 What is the uncertainty over a mortality model?

Q2 How does this feed into uncertainty over the liabilities?

2 Estimation and likelihood

2 Estimation by likelihood

- Specify a statistical model with parameter $\boldsymbol{\theta}$.
- Form the log-likelihood function, $\ell(\boldsymbol{\theta})$.
- Find $\hat{\boldsymbol{\theta}}$ which maximises $\ell(\boldsymbol{\theta})$.
- $\hat{\boldsymbol{\theta}}$ is the maximum-likelihood estimate (MLE) of unknown true value $\boldsymbol{\theta}^*$.

2 $\hat{\boldsymbol{\theta}}$ in Longevitas

$\hat{\boldsymbol{\theta}}$ is the Estimate column on the **Model** tab:

Model | Parameters | Chart | Curves | Covariance | Data

Model Overview

Model Parameters

Parameter	Estimate	Standard error	Z-value	Pr(>|z|)	Sig.	Lives	Decrements
AgeGradientYoungest	-4.28717	0.549765	-7.8	0	***	84389	19435
Gender.F	-0.95287	0.041431	-23	0	***	51064	10690
Gender.F:Oldest	-0.159249	0.032658	-4.88	0	***	51064	10690
Intercept	-3.97482	0.10836	-36.68	0	***	84389	19435
Oldest	-0.769237	0.033451	-23	0	***	84389	19435
PensionSize.2	-0.171969	0.035264	-4.88	0	***	20256	4662
PensionSize.3	-0.899687	0.056532	-15.91	0	***	10116	1502

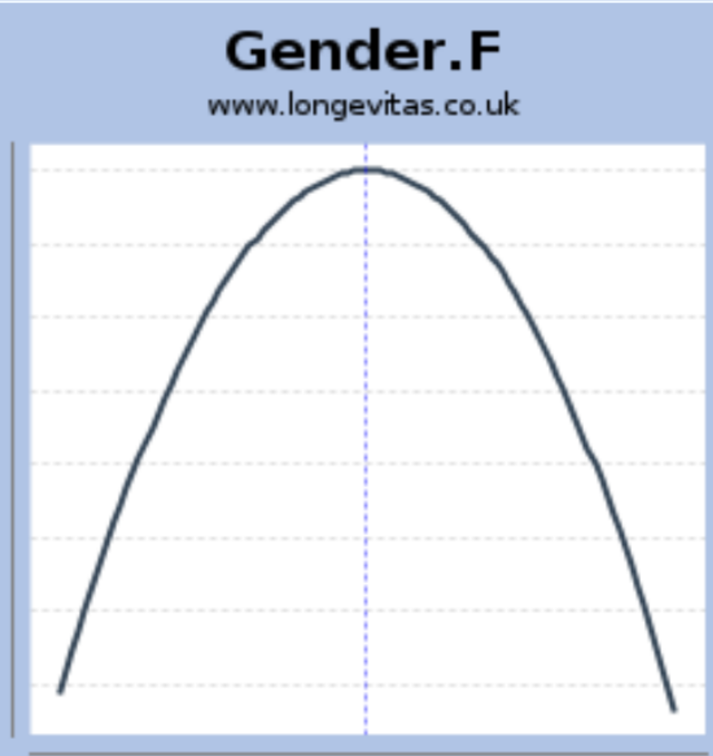
2 Likelihoods in Longevitas

Profile log-likelihoods are in the **Parameters** tab:

Model | Parameters | Chart | Curves | Covariance | Data

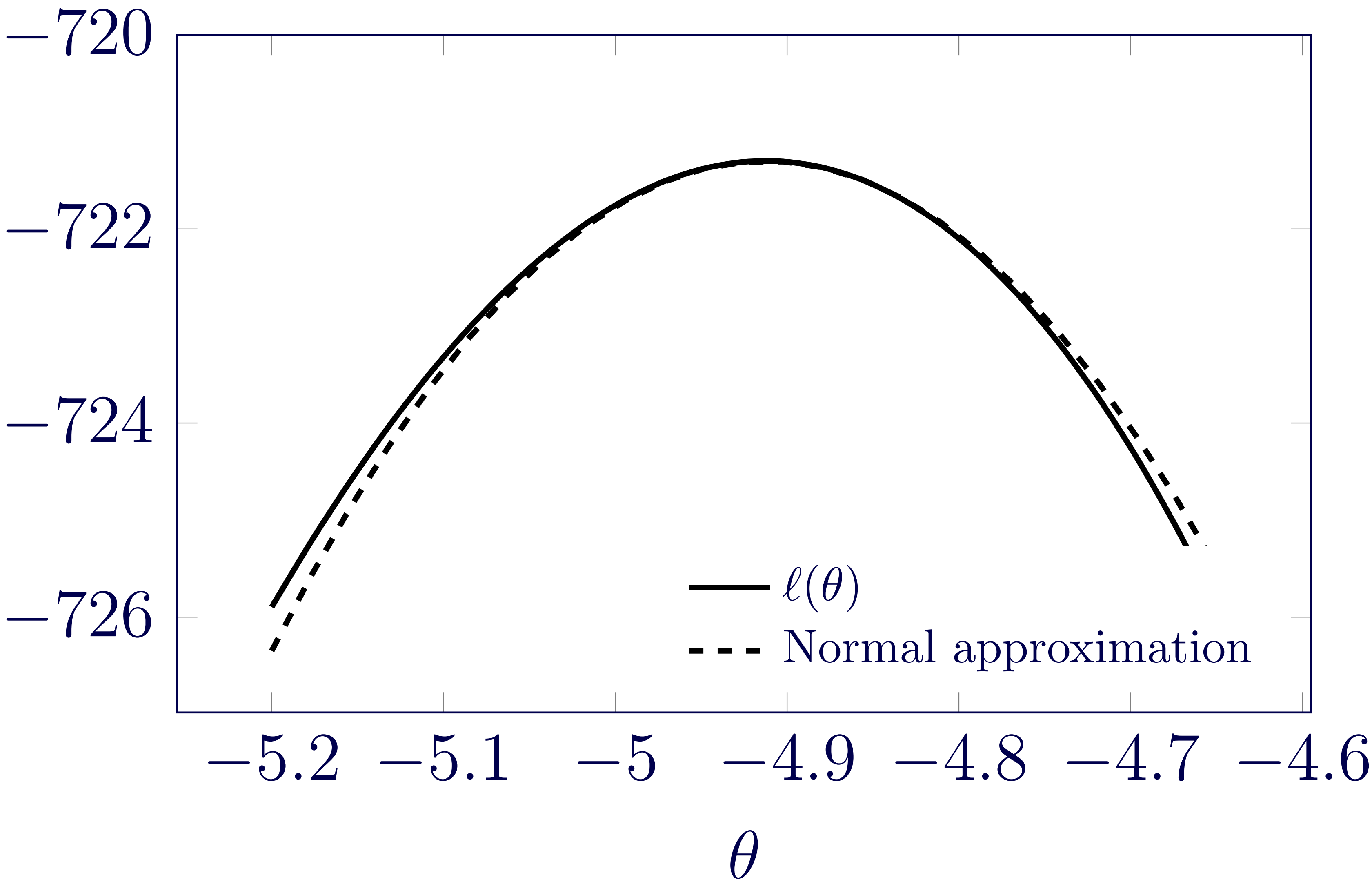
Explore: Gender.F | Plot: Partial log-likelihood | Using: Spline chart | Go

Partial log-likelihood for Gender.F



Parameter attributes	
Estimate	-0.95287
Parameter type	This parameter is a main effect and does not vary by age, duration or time.
Standard error	0.0414
Z-value	-23
p-value	0
Significance	***
Coefficient of variation	-4.35%
Exposure	344110
Decrements	10690
Lives	51064

2 $\ell(\theta)$ in one dimension



Source: Richards [2016, Figure 1].

2 Maximum likelihood theorem

- $\hat{\boldsymbol{\theta}}$ has multivariate normal distribution (MVN)†.
- This holds *irrespective of the model structure.*
- $\hat{\boldsymbol{\theta}} \sim \mathrm{MVN}(\boldsymbol{\theta}^*, \boldsymbol{\Sigma}^*)$.
- $\boldsymbol{\theta}^*$ and $\boldsymbol{\Sigma}^*$ unknown, so use $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\Sigma}}$.
- $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\Sigma}}$ both estimated from log-likelihood.

† Cox and Hinkley [1996, Chapter 9(iii)].

2 $\hat{\Sigma}$ in Longevitas

$\hat{\Sigma}$ is the covariance matrix on the **Covariances** tab:

Model | Parameters | Chart | Curves | Covariance | Data

Model Overview

Correlations

Covariances

	AgeGradientYoungest	Gender.F	Gender.F:Oldest	Intercept	Oldest	PensionSize.2	PensionSize.3
AgeGradientYoungest	0.302242	-0.00239569	0.00326321	-0.0568288	-0.011784	0.000487892	0.00209703
Gender.F	-0.00239569	0.00171656	-0.000881487	-0.000492527	0.000594854	0.000260095	0.000405566
Gender.F:Oldest	0.00326321	-0.000881487	0.00106656	-0.000105426	-0.000786624	-3.19636e-05	2.63637e-07
Intercept	-0.0568288	-0.000492527	-0.000105426	0.0117419	0.00170574	-0.000603609	-0.000990392
Oldest	-0.011784	0.000594854	-0.000786624	0.00170574	0.00111897	2.20406e-05	-5.63628e-05
PensionSize.2	0.000487892	0.000260095	-3.19636e-05	-0.000603609	2.20406e-05	0.00124358	0.000448612
PensionSize.3	0.00209703	0.000405566	2.63637e-07	-0.000990392	-5.63628e-05	0.000448612	0.00319584

2 $\hat{\Sigma}$ in Longevitas

- Covariance of variable with itself is its variance.
- Variance of Gender.F is 0.00171656.
- Standard error of Gender.F is $\sqrt{0.00171656} = 0.041431$

Standard errors in Longevitas:

Model | Parameters | Chart | Curves | Covariance | Data

Model Overview

Model Parameters

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Gender.F:Oldest	-0.159249	0.032658	-4.88	0	***	51064	10690
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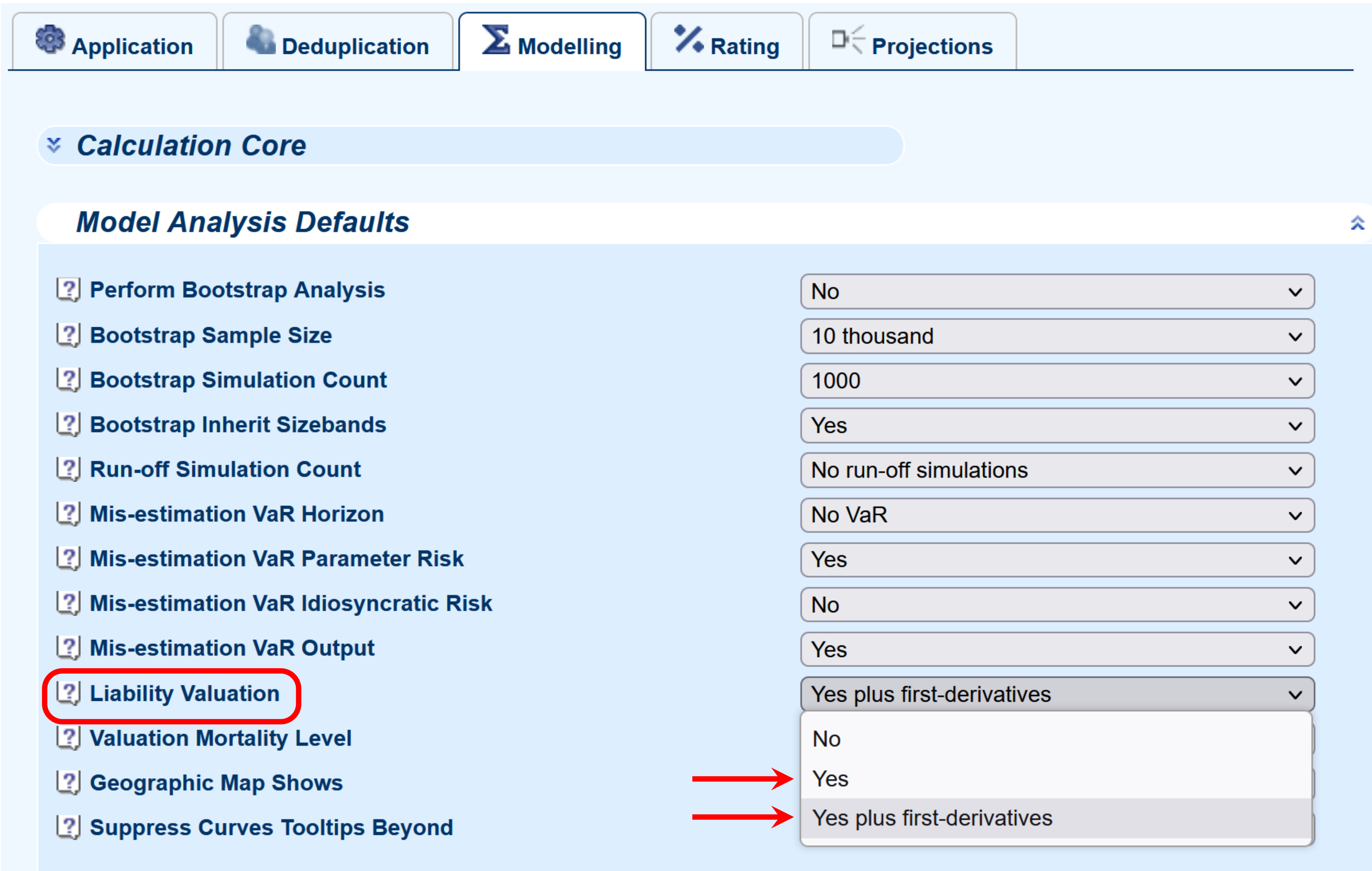
PensionSize.3	-0.899687	0.056532	-15.91	0	***	10116	1502

3 Liability valuation

3 Valuation using $\hat{\theta}$

Enable valuation under **Settings→Configuration**:



3 Valuation in **Model** tab

LONGEVITAS

Model | Parameters | Chart | Curves | Covariance | Data

Model Overview

Model Parameters

Valuation

Valuation Setting	Value	
Liability Type	annuity	
Records Used	survivors only	
Valuation Field	Amount	
Interest	0.00	
	Males	**Females**
Lives	24580	40374
Amounts At Outset	181486039.56	142874709.68

Payment basis: continuous

Payment basis: annual in arrears

Payment basis: annual in advance

3 Reserve estimate

LONGEVITAS

Valuation

Valuation Setting	Value	
Liability Type	annuity	
Records Used	survivors only	
Valuation Field	Amount	
Interest	0.00	
	Males	**Females**
Lives	24580	40374
Amounts At Outset	181486039.56	142874709.68

Payment basis: continuous

	Males	Females
Time lived	323269.1	616150.8
Reserves	2668712705.97	2289494200.05
Standard error	17495191.25	9978918.47
Coeff. of variation	0.66%	0.44%

Derivatives

4 Valuation uncertainty

4 Valuation uncertainty

- $\hat{\boldsymbol{\theta}}$ is uncertain.
- So reserve, $V(\hat{\boldsymbol{\theta}})$, is also uncertain.
- Use delta method to approximate variance of reserve.

4 Delta method

- $\hat{\boldsymbol{\theta}}$ is a random variable, i.e. $\hat{\boldsymbol{\theta}} \sim \mathrm{MVN}(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\Sigma}})$.
- Reserve, $V(\hat{\boldsymbol{\theta}})$, is a function of a random variable.
- $\mathrm{Var}[V(\hat{\boldsymbol{\theta}})] \approx \boldsymbol{a}^T \hat{\boldsymbol{\Sigma}} \boldsymbol{a}$, where $\boldsymbol{a} = \left.\dfrac{\partial V(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}}$

4 *a* in **Derivatives** section

Valuation

Valuation Setting	Value	
Liability Type	annuity	
Records Used	survivors only	
Valuation Field	Amount	
Interest	0.00	
	Males	**Females**
Lives	24580	40374
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Derivatives

4 $\boldsymbol{a}$ in **Derivatives** section

$\boldsymbol{a}$ for males, females and combined:

Derivatives

Below are the first partial derivatives of the liability values with respect to the various parameters, i.e. the sensitivities of the liability values to change in each parameter value. These are used in conjunction with the parameter covariance matrix to approximate the variances of the liability values using the delta method, and thus obtain the standard errors of the liability values shown in the table above.

With respect to:	Males	Females	Combined
AgeGradientYoungest	-112676207.83	-76925796.87	-189602004.70
Gender.F	0.00	-303831771.67	-303831771.67
Gender.F:Oldest	0.00	-583674667.03	-583674667.03
Intercept	-460200231.77	-303831771.67	-764032003.45
Oldest	-674637060.04	-583674667.03	-1258311727.07
PensionSize.2	-131775201.53	-114406787.98	-246181989.51
PensionSize.3	-274551385.60	-76420216.09	-350971601.70

4 Reserve standard error

Valuation

Valuation Setting	Value	
Liability Type	annuity	
Records Used	survivors only	
Valuation Field	Amount	
Interest	0.00	
	Males	**Females**
Lives	24580	40374
Amounts At Outset	181486039.56	142874709.68

Payment basis: continuous

	Males	Females
Time lived	323269.1	616150.8
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Standard error	17495191.25	9978918.47
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Derivatives

5 Conclusions

5 Conclusions

- Maximum-likelihood estimation produces $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\Sigma}}$.
- Estimate of reserves using $\hat{\boldsymbol{\theta}}$.
- Estimate variance of reserves using delta method.
- Implicit assumption of normal distribution for reserve valuation.

References I

D. R. Cox and D. V. Hinkley. *Theoretical Statistics*. Chapman and Hall, 1996. ISBN 0-412-16160-5.

S. J. Richards. Mis-estimation risk: measurement and impact. *British Actuarial Journal*, 21 (3):429–457, 2016. doi: 10.1017/S1357321716000040.

More on mis-estimation risk at www.longevitas.co.uk/mis-estimation-risk

6 Legal matters

Longevitas is a registered trademark:

- in the UK (No. 2434941),
- in the USA (No. 3707314), and
- in the European Union (No. 5854518).

Longevitas client webinar

Mis-estimation for pricing

Stephen J. Richards

26th January 2023

Overview

1. Motivation

2. Recap

3. Reserve distribution

4. Delta method v. sampling

5. Conclusions

About this presentation

This presentation is about how to use the mis-estimation features in the Longevitas survival-modelling software.

For the theory and technical details, see

www.longevitas.co.uk/mis-estimation-risk

1 Motivation

1 Motivation

What is the true distribution of reserve values in response to parameter uncertainty?

2 Recap

2 Recap

- $\hat{\boldsymbol{\theta}}$ is the maximum-likelihood estimate (MLE).
- $\hat{\boldsymbol{\theta}}$ has approximate MVN$(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\Sigma}})$ distribution.
- Reserve, $V(\hat{\boldsymbol{\theta}})$, is thus also a random variable.
- Approximate variance of $V(\hat{\boldsymbol{\theta}})$ using delta method.

2 Problem

However:

- Using delta method assumes that $V(\hat{\boldsymbol{\theta}})$ is normally distributed.
- In practice $V(\hat{\boldsymbol{\theta}})$ is not necessarily normally distributed.
- Want to know the true distribution of $V(\hat{\boldsymbol{\theta}})$, *especially if not symmetric.*

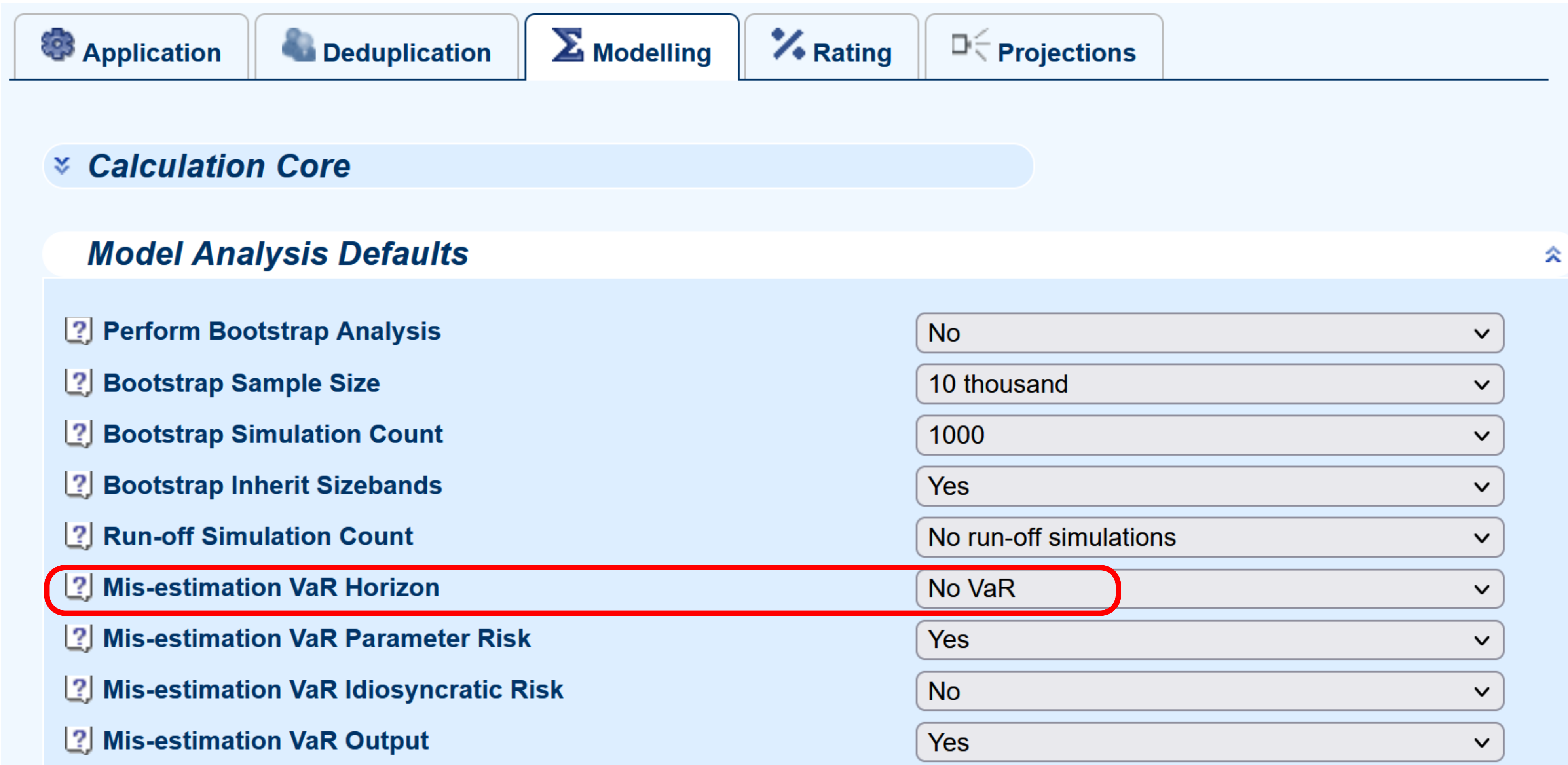
3 Reserve distribution

3 Reserve distribution

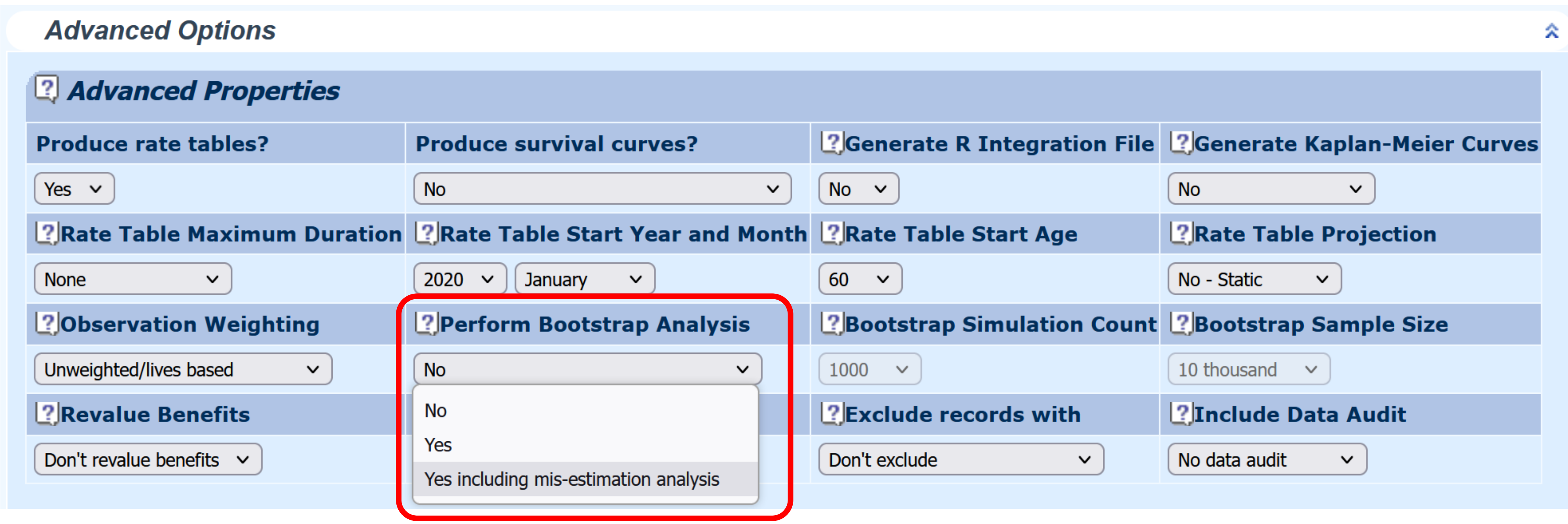
Sampling approach from Richards [2016]:

1. Sample $\boldsymbol{\theta}_j$ from $\text{MVN}(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\Sigma}})$.
2. Calculate $V(\boldsymbol{\theta}_j)$.
3. Repeat steps 1 and 2 for $j = 1, 2, \ldots,$ `nsamp`.
4. Resulting set of `nsamp` values represents sample from true distribution of $V(\hat{\boldsymbol{\theta}})$.

Ensure the value-at-risk option is switched off:

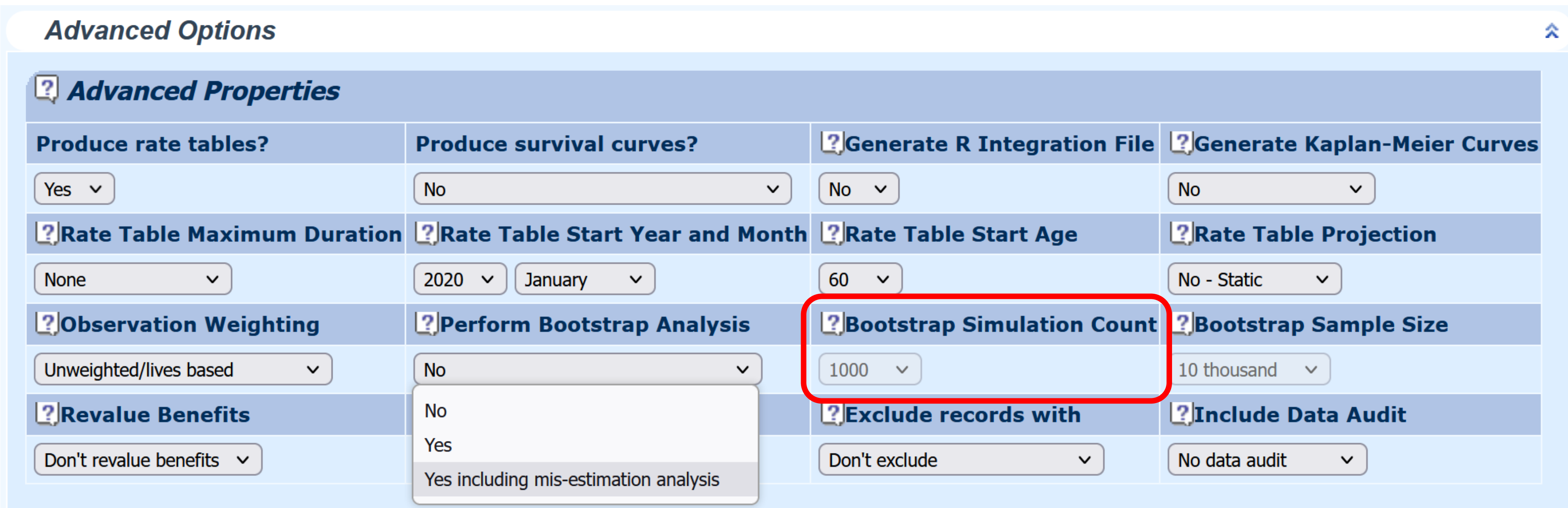


Specify mis-estimation on first modelling screen:



3 Mis-estimation simulations

Specify `nsamp` for mis-estimation (and bootstrapping):



3 Mis-estimation outputs

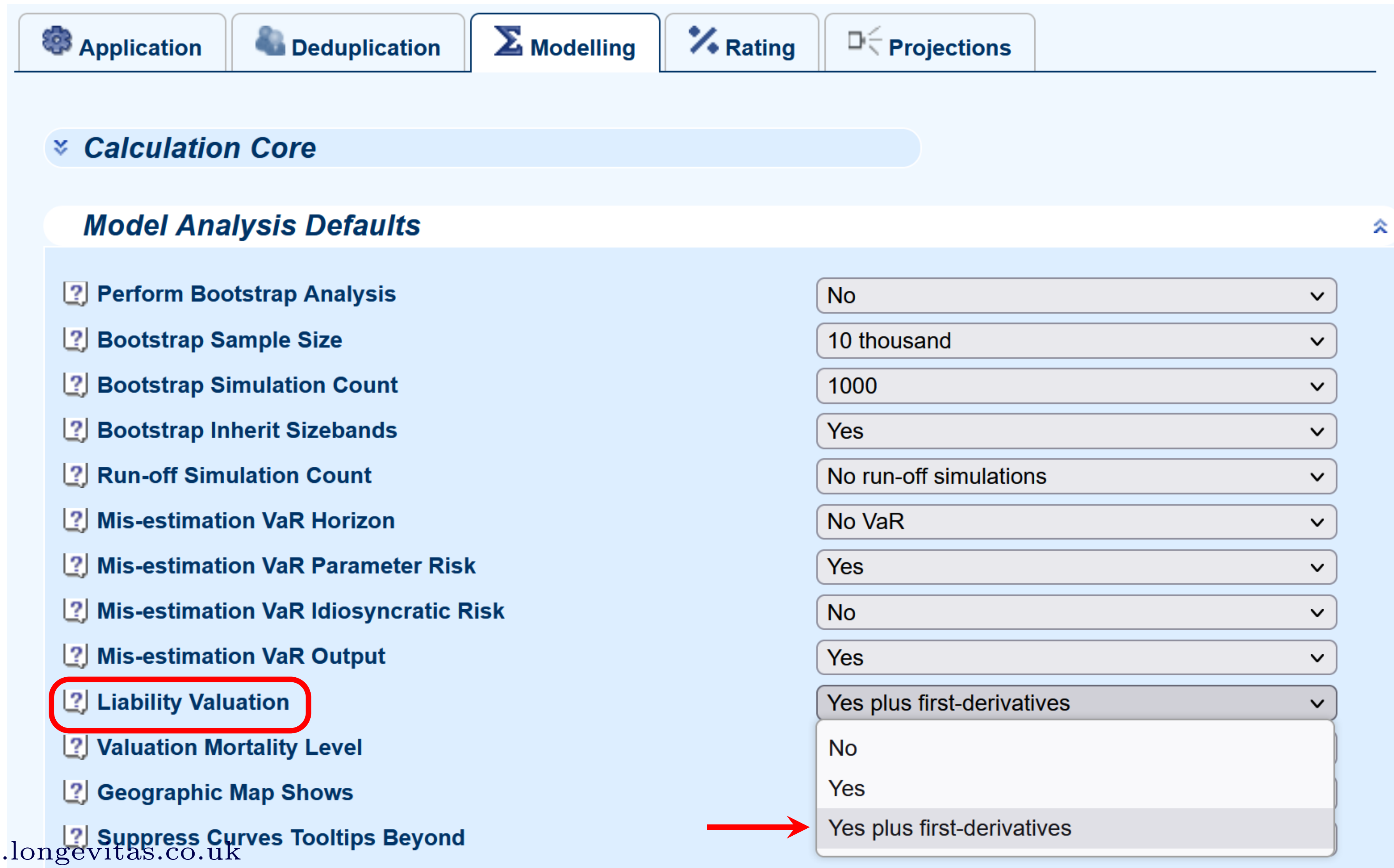
Mis-estimation and bootstrap results appear together:

Time	Name	Text
12:30:05	**Model 79711**	Survival(Hermite-II) - For mis-estimation presentation with 10000 simulations (AgeGradientYoungest, Gender, Oldest, PensionSize, Gender:Oldest)
12:41:11	Decrement: DEATH	Model Input XML Rate Tables Residuals Bootstrap Input Bootstrap Results Mis-Estimate Pulse

4 Delta method v. sampling

4 Delta method

Ask for delta method by enabling valuation derivatives under **Settings→Configuration**:



4 Delta method

Results of delta method under continuous valuation:

Valuation

Valuation Setting	Value	
Liability Type	annuity	
Records Used	survivors only	
Valuation Field	Amount	
Interest	0.00	
	Males	Females
Lives	24580	40374
Amounts At Outset	181486039.56	142874709.68

Payment basis: continuous

	Males	Females
Time lived	323269.1	616150.8
Reserves	2668712425.59	2289494139.52
Standard error	17495188.42	9978917.02
Coeff. of variation	0.66%	0.44%

4 Delta method

	Males	Females
Reserve (£m)	2,669	2,289
Standard error (£m)	17.5	10.0

4 Sampling approach

Use mis-estimation output file:

Time	Name	Text
12:30:05	**Model 79711**	Survival(Hermite-II) - For mis-estimation presentation with 10000 simulations (AgeGradientYoungest, Gender, Oldest, PensionSize, Gender:Oldest)
12:41:11	Decrement: DEATH	Model · Input · XML · Rate Tables · Residuals · Bootstrap Input · Bootstrap Results · Mis-Estimate · Pulse

Use mis-estimation output file:

	E	G	H	I	J	K	L	M	N
1	ExpectedPresentValueMalesContinuous	ExpectedPresentValueFemalesContinuous	AgeGradientYoungest	Gender.F	Gender.F:Oldest	Intercept	Oldest	PensionSize.2	PensionSize.3
2	2651025876	2273760717	-4.265692	-0.938834	-0.15497	-4.013486	-0.736687	-0.209652	-0.840957
3	2661679737	2282490174	-4.272845	-0.928793	-0.158497	-3.967263	-0.778794	-0.184164	-0.863494
4	2669948989	2281460464	-4.112005	-0.884203	-0.181287	-4.027025	-0.772404	-0.137001	-0.897609
5	2659892578	2294741595	-4.960897	-1.045544	-0.129697	-3.794761	-0.772422	-0.126729	-0.907336
6	2682378094	2292899433	-4.646145	-0.945333	-0.175459	-3.9246	-0.735257	-0.15079	-0.980943
7	2651949887	2292317999	-4.774853	-0.937442	-0.196579	-3.887774	-0.734145	-0.138568	-0.886267
8	2670809057	2292162814	-4.370889	-0.973476	-0.143136	-3.962309	-0.771169	-0.208992	-0.871526
9	2674025403	2288312857	-4.430025	-0.951354	-0.139202	-3.977542	-0.77802	-0.159694	-0.840199
10	2705245813	2301477149	-4.189709	-0.940179	-0.144941	-3.94713	-0.815461	-0.236753	-0.973414
11	2678041018	2289727971	-4.117512	-0.968994	-0.139727	-3.972671	-0.804764	-0.148893	-0.930393
12	2682246468	2291790178	-4.272211	-0.945537	-0.166958	-3.971986	-0.75328	-0.220268	-0.976469
13	2652937560	2275476647	-5.161946	-0.905199	-0.199297	-3.852893	-0.684524	-0.127817	-0.915487
14	2701471438	2279742590	-3.362895	-0.997768	-0.074036	-4.155678	-0.834679	-0.194606	-0.922094
15	2654323606	2263164239	-4.947679	-0.950599	-0.132107	-3.811085	-0.746952	-0.190782	-0.89645
16	2691533412	2291896456	-4.47731	-0.996029	-0.109789	-3.908488	-0.800212	-0.205467	-0.923186
17	2689040769	2278902984	-3.078666	-0.928135	-0.121823	-4.224058	-0.833462	-0.160286	-0.898785
18	2665796369	2296425876	-4.773069	-0.960412	-0.178985	-3.905613	-0.726283	-0.185232	-0.9046
19	2686445796	2300776455	-4.303641	-1.000535	-0.132903	-3.95102	-0.797902	-0.16415	-0.930354
20	2641987821	2287310003	-3.723254	-0.922005	-0.218019	-4.098357	-0.734231	-0.149375	-0.922792
21	2675367554	2298341852	-3.559969	-0.984055	-0.160583	-4.118105	-0.786914	-0.142549	-0.953192
22	2674225077	2306171008	-4.371478	-0.965088	-0.176479	-3.950686	-0.769861	-0.191047	-0.914935
23	2673069282	2288977112	-4.870113	-0.962771	-0.145316	-3.871513	-0.761405	-0.147224	-0.880657
24	2652148254	2277901993	-4.39163	-0.886525	-0.191128	-4.005481	-0.72601	-0.192582	-0.840928

4 Sampling approach

	Males	Females
Reserve (£m)	2,669	2,290
Standard error (£m)	17.4	9.9

4 Ratio of two approaches

	Males	Females
Reserve	100.00%	99.99%
Standard error	100.61%	99.80%

5 Conclusions

5 Conclusions

- In this particular case, the delta method was a highly accurate approximation.
- This may not be true of all models and data sets.
- Use either delta method or sampling mis-estimation to understand response of reserve estimate to parameter uncertainty.

References I

S. J. Richards. Mis-estimation risk: measurement and impact. *British Actuarial Journal*, 21 (3):429–457, 2016. doi: 10.1017/S1357321716000040.

More on mis-estimation risk at www.longevitas.co.uk/mis-estimation-risk

6 Legal matters

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Longevitas client webinar

Mis-estimation: value-at-risk

Stephen J. Richards

26th January 2023

LONGEVITAS™

Overview

1. Motivation
2. Pricing view v. value-at-risk
3. Simulation
4. Conclusions

About this presentation

This presentation is about how to use the mis-estimation features in the Longevitas survival-modelling software.

For the theory and technical details, see

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1 Motivation

1 Mis-estimation: value-at-risk

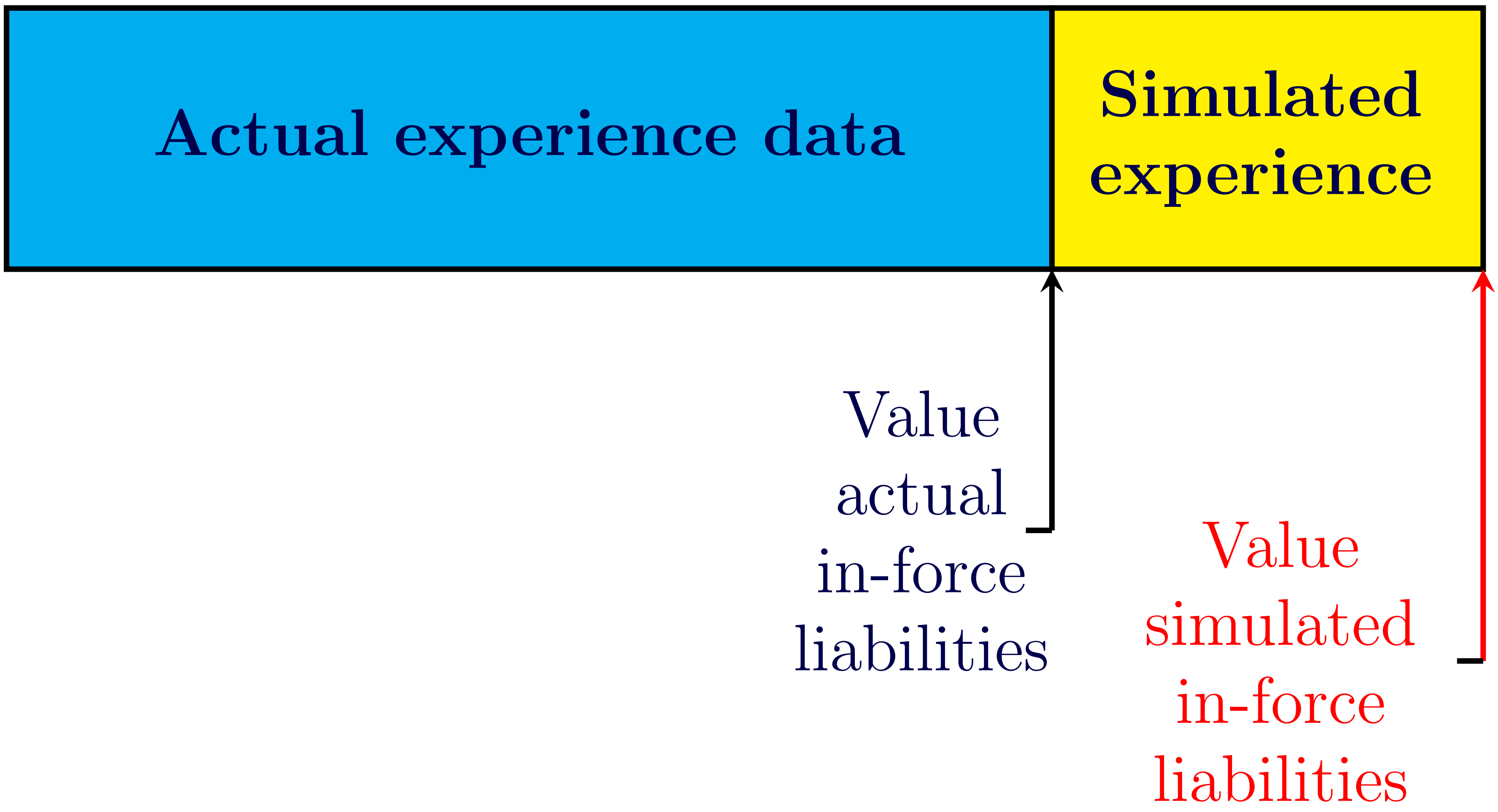
Q1 How sensitive is the mortality model to new data?

Q2 How sensitive is the reserve to model recalibration?

Q3 What capital is needed to cover recalibration risk?

2 Pricing view v. value-at-risk

2 Pricing v. VaR



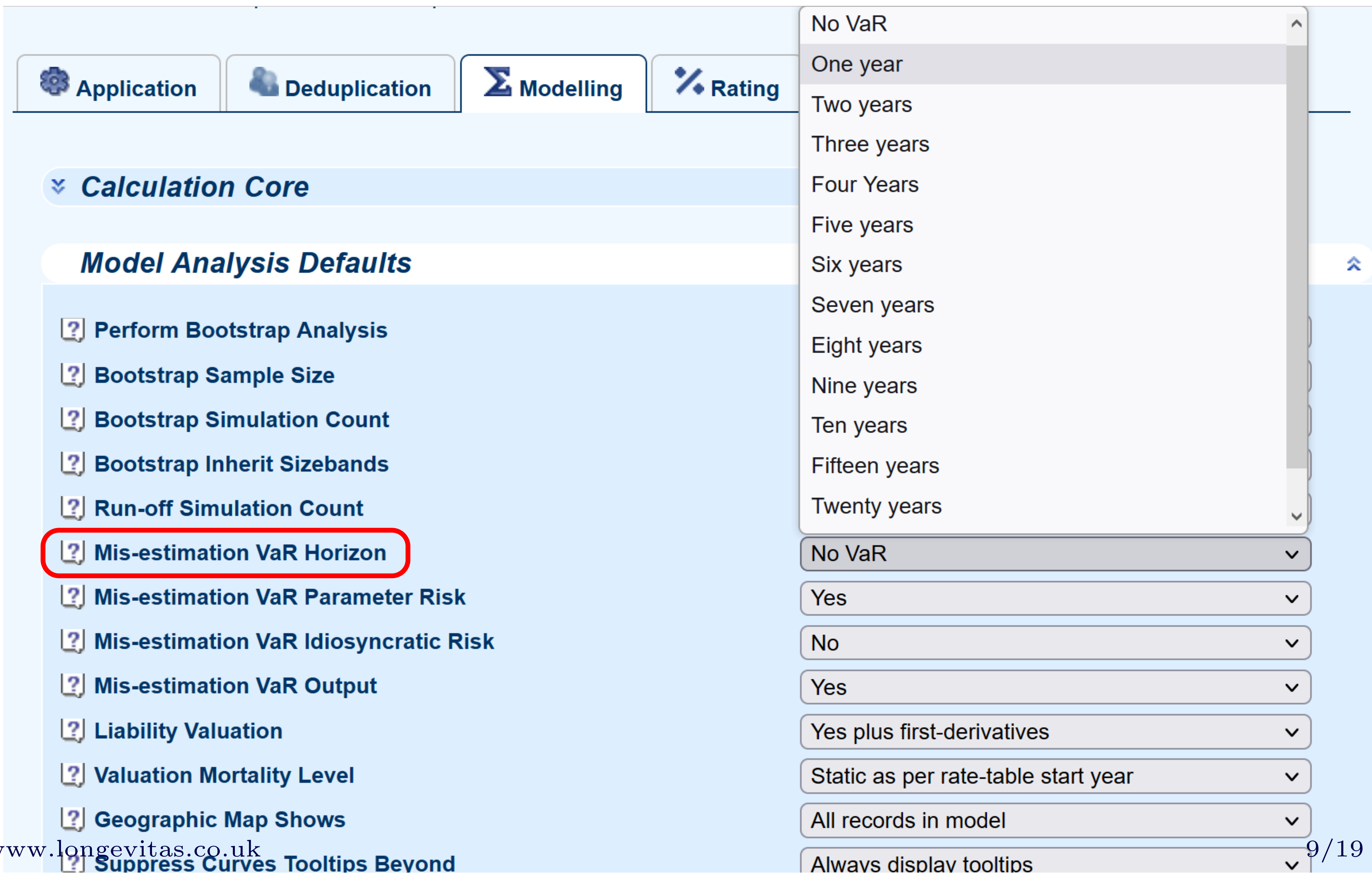
2 Liability valuation

Q. Why value liabilities at end of simulation period?

A. To simultaneously measure mis-estimation risk and idiosyncratic risk; see Richards [2021, Table 9].

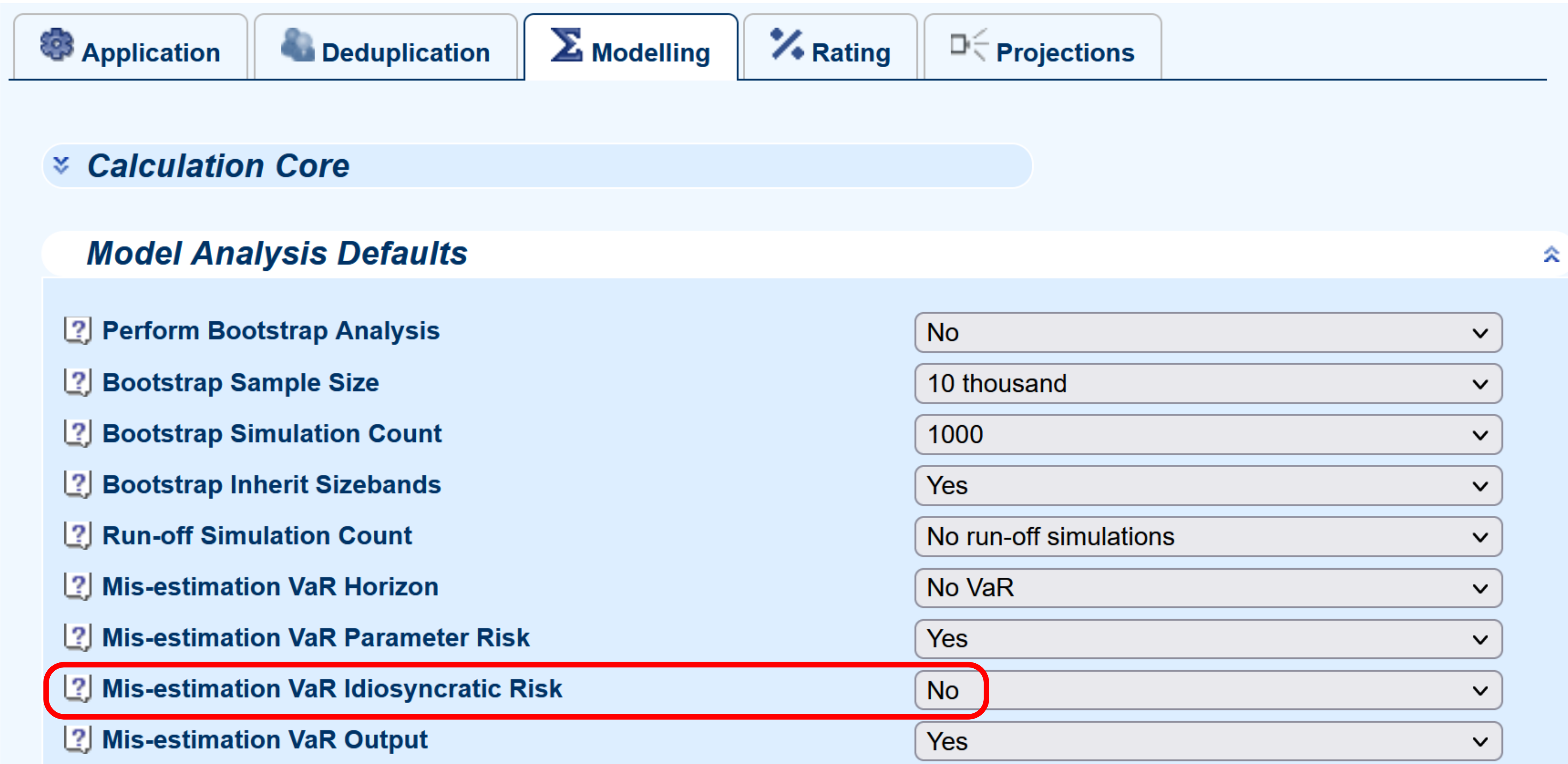
2 VaR mis-estimation

Switch on VaR mis-estimation by selecting horizon:



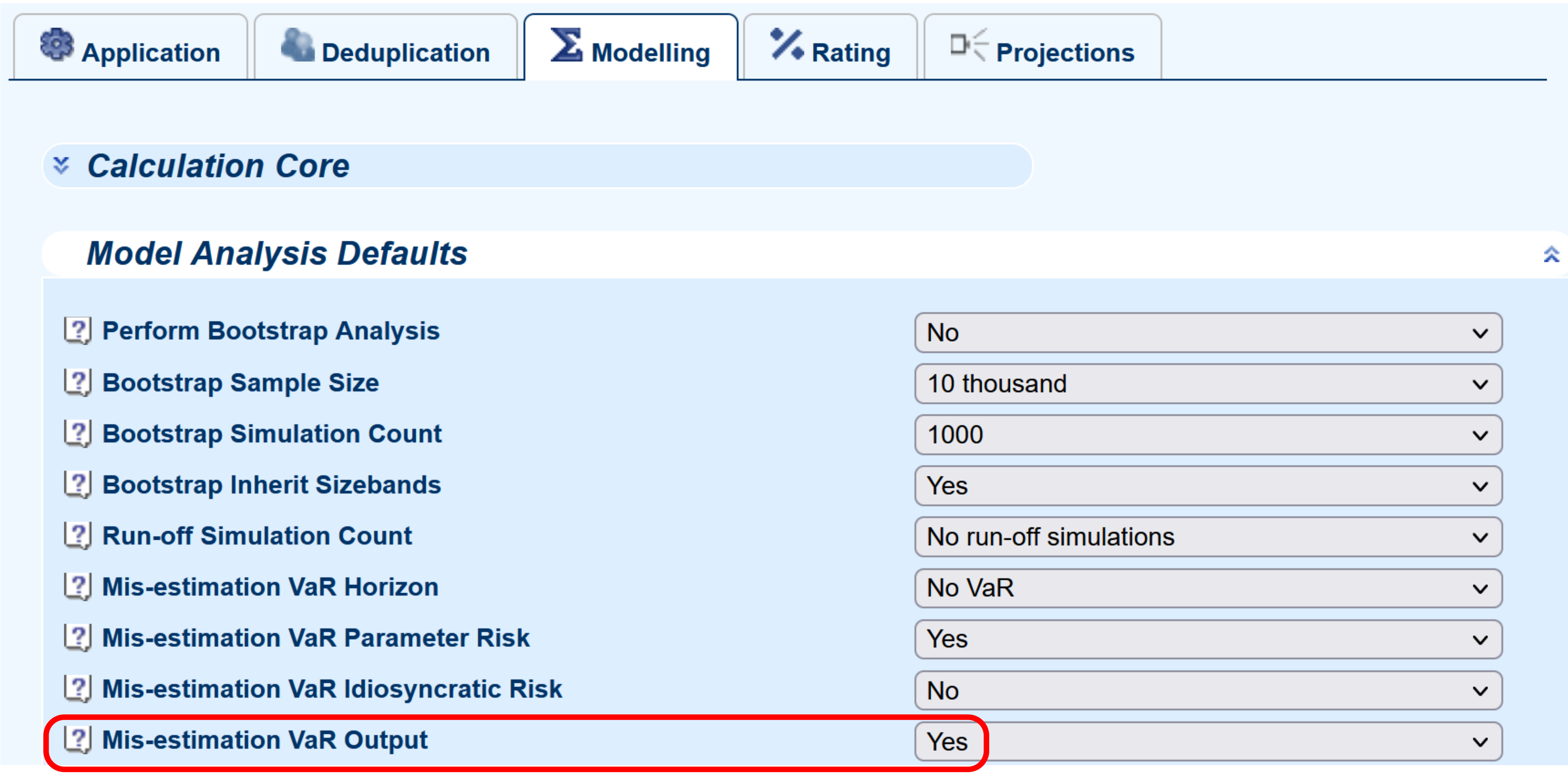
2 Liability valuation

Switch between actual in-force and simulated in-force:



2 Idiosyncratic risk

Switch on output of simulated deaths and time lived:



3 Simulation

3 Recap

- $\hat{\boldsymbol{\theta}}$ is the maximum-likelihood estimate (MLE).
- $\hat{\boldsymbol{\theta}}$ has approximate MVN$(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\Sigma}})$ distribution.

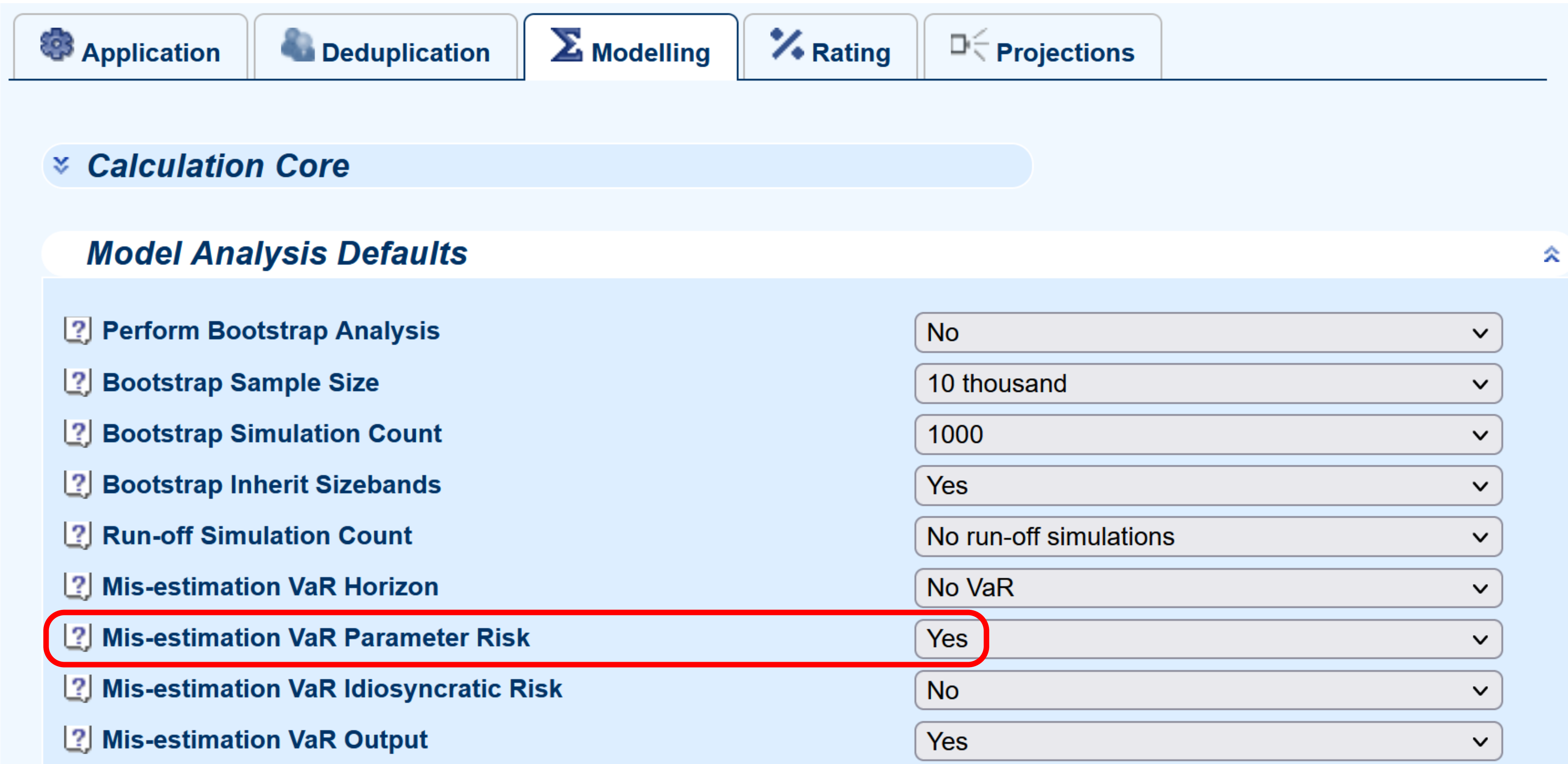
3 Simulation for VaR

Two options for simulating n-year experience:

1. Without parameter risk: just use $\hat{\boldsymbol{\theta}}$.
2. With parameter risk: sample from MVN$(\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\Sigma}})$.

3 Parameter risk

Switch VaR parameter risk on or off in configuration:



4 Conclusions

4 Conclusions

Options for liability estimation risk in Longevitas:

1. **Delta method**. For pricing block transactions. Assumes normal distribution for liability value, which might not always be true.
2. **Richards [2016]**. For pricing block transactions. Makes no assumption about distribution of liability value.
3. **Richards [2021]**. Value-at-risk approach for one-year Solvency II reporting, ORSA etc.

References I

S. J. Richards. Mis-estimation risk: measurement and impact. *British Actuarial Journal*, 21 (3):429–457, 2016. doi: 10.1017/S1357321716000040.

S. J. Richards. A value-at-risk approach to mis-estimation risk. *British Actuarial Journal*, page to appear, 2021. doi: 10.1017/S1357321721000131.

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