

# Robust mortality forecasting in the presence of outliers

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27th November 2023



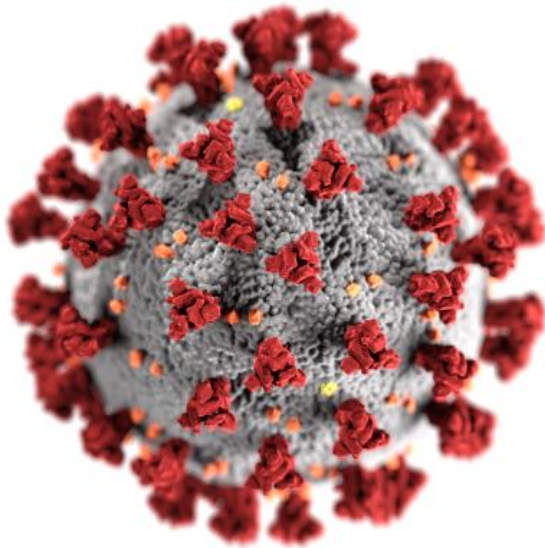
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1. Motivation
2. Some alternative methods
3. Univariate forecasts
4. Multivariate forecasts
5. 2D  $P$ -spline model
6. Conclusions

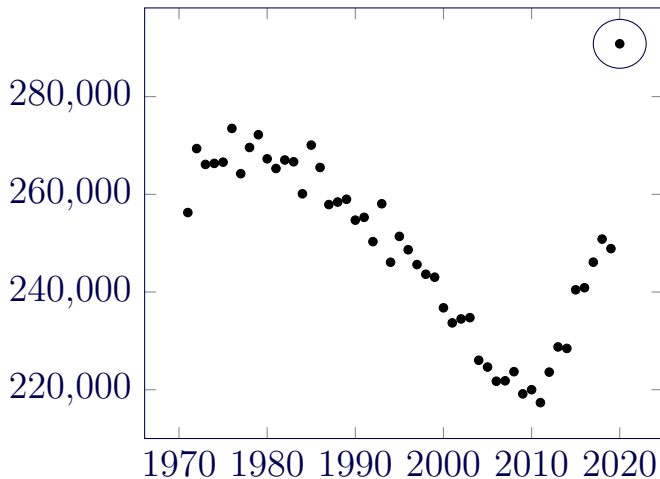
# 1 Motivation



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# 1 Male deaths, England & Wales

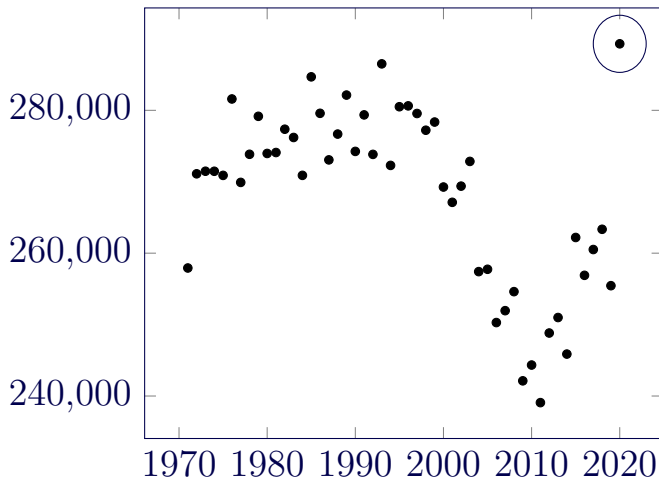


Source: Richards [2024, Figure 1(b)].

# 1 Female deaths, England & Wales



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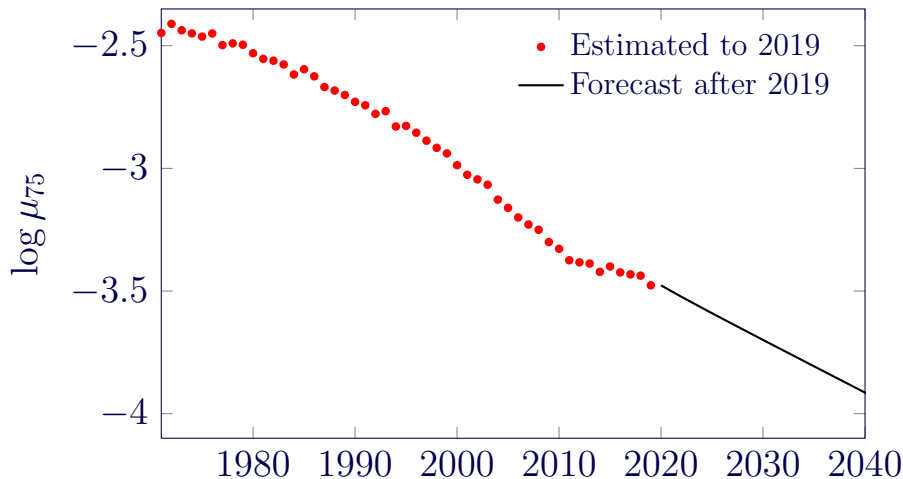
Source: Richards [2024, Figure 1(d)].

## Outliers:

1. Distort forecasts.
2. Biase forecast starting points.
3. Inflate variance, and thus VaR capital.

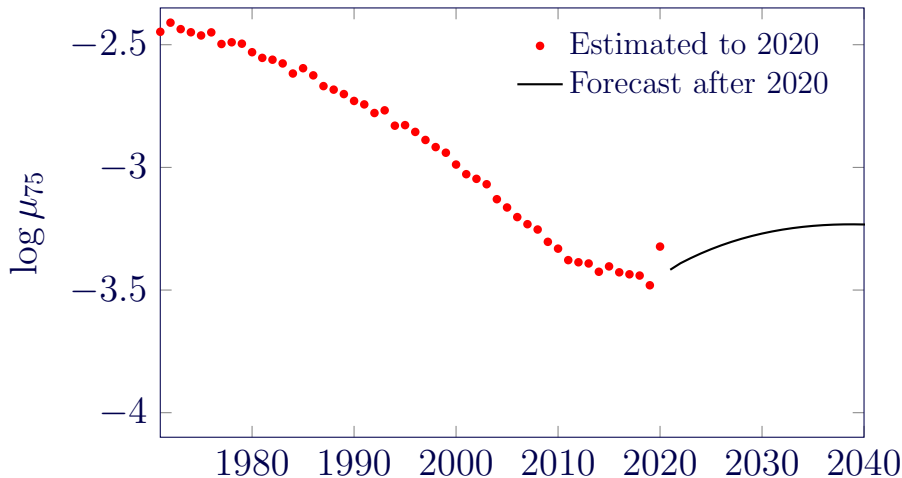
Covid-19 distorts central projections...

## ARIMA forecast of time index in Lee-Carter model:



Source: Data for males in England & Wales, ages 50–105, 1971–2019.

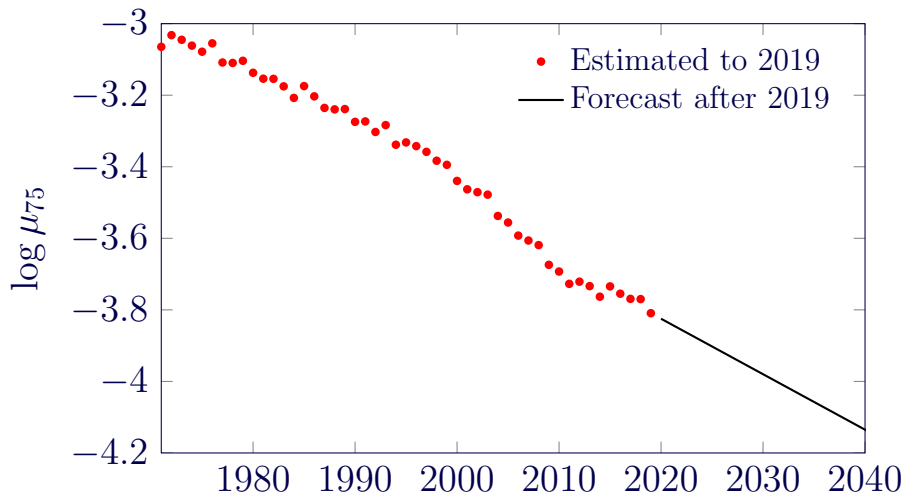
## ARIMA forecast of time index in Lee-Carter model:



Source: Data for males in England & Wales, ages 50–105, 1971–2020.

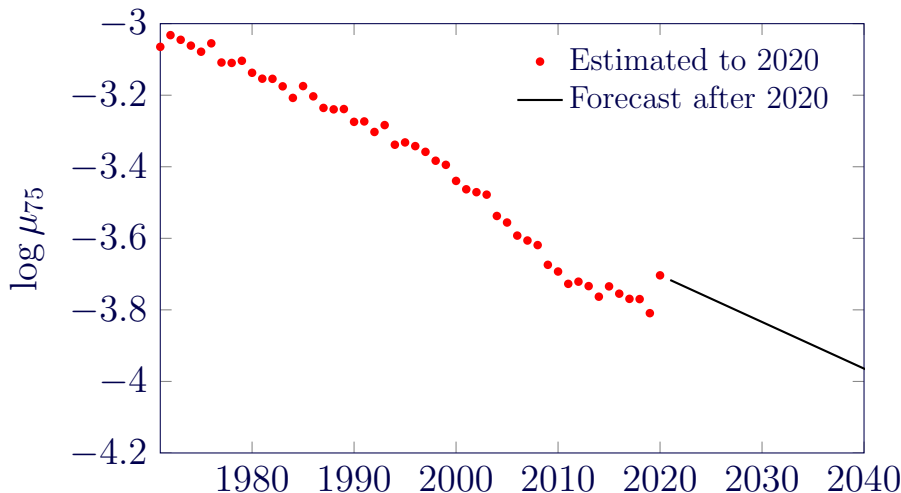
Covid-19 leads to biased starting points...

## Bivariate random-walk forecast under M5 model:



Source: Data for females in England & Wales, ages 60–105, 1971–2019.

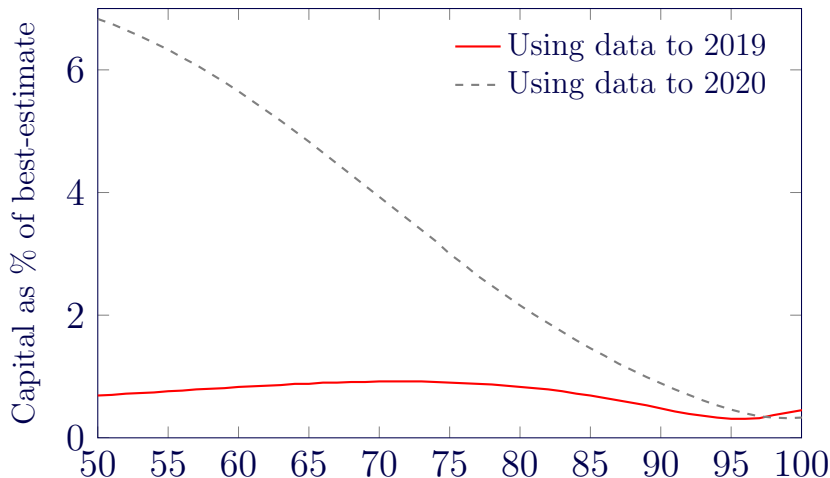
## Bivariate random-walk forecast under M5 model:



Source: Data for females in England & Wales, ages 60–105, 1971–2020.

Outliers increase VaR capital requirements...

# Value-at-risk capital at 99.5%



Source: 10,000 recalibrations of Lee-Carter model using data for males in England & Wales. Annuity cashflows discounted at 0% per annum.

- Covid-19 breaks forecasting models in three important ways.
- How can we robustify forecasts for actuarial tasks?

1. Remove distortion in parameter estimates.
2. Calculate “clean” starting points for forecasts.
3. Estimate variance robustly.
4. Need objective methodology for (1)-(3) to allow repeated recalibration under VaR-style simulations.

- Identify outliers with statistical tests.
- Co-estimate outlier effects with other parameters.

## 2 Some alternative methods

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## 2 Ignoring affected data



- Workable temporary solution just to use data to 2019.
- ✗ No longer workable in 2023.
- ✗ Affected data no longer at end of data series.

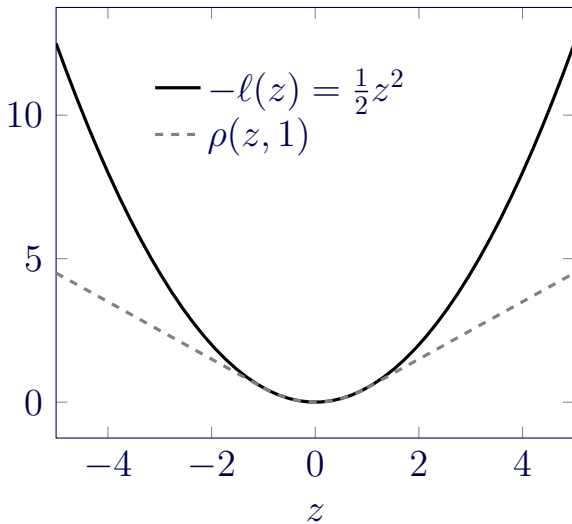
- ✗ Subjective in identifying outlier location.
- ✗ Weights set arbitrarily.
- ✗ Does not estimate outlier effects.
- ✗ Tells you nothing about the nature of the outlier.
- ✗ Doesn't fully protect log-likelihood against distortion and bias.

$$-\ell(z) \propto \frac{1}{2}z^2 \quad (1)$$

- Called loss function.
- Contribution to normal log-likelihood is quadratic.
- Outliers distort quadratically.

Early attempts at robustification replaced  $-\ell$  with a function,  $\rho$ , that behaved similarly for non-outliers, but which reduced or even eliminated the contribution from outliers. See Martin et al. [1983].

## 2 Negative log-likelihood



Source: Richards [2024, Figure 2].

# 3 Univariate forecasts

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- A univariate model has multiple parameter vectors, but only one represents a time index to forecast.
- Example from Lee and Carter [1992]:

$$\log \mu_{x,y} = \alpha_x + \beta_x \kappa_y \quad (3)$$

- $\hat{\alpha}_x$  and  $\hat{\beta}_x$  are held constant in the forecast.
- An ARIMA model is fitted to the  $\hat{\kappa}_y$  time index to forecast the trend.

## Outlier

An observation that is further from the one-year-ahead forecast than is consistent with the noise variance.

# 3 Four outlier types



- IO** Innovation outlier
- AO** Additive outlier
- TC** Temporary change
- LS** Level shift

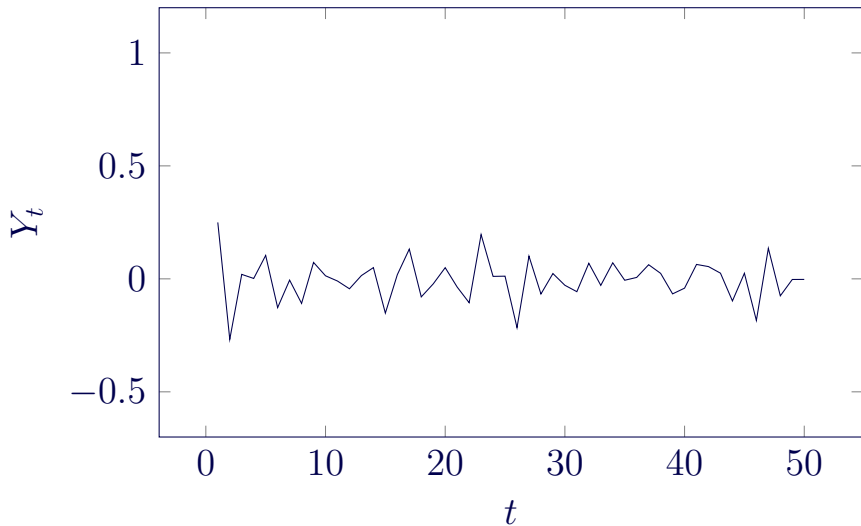
# 3 Illustrating outlier types



Consider a moving-average (MA) process:

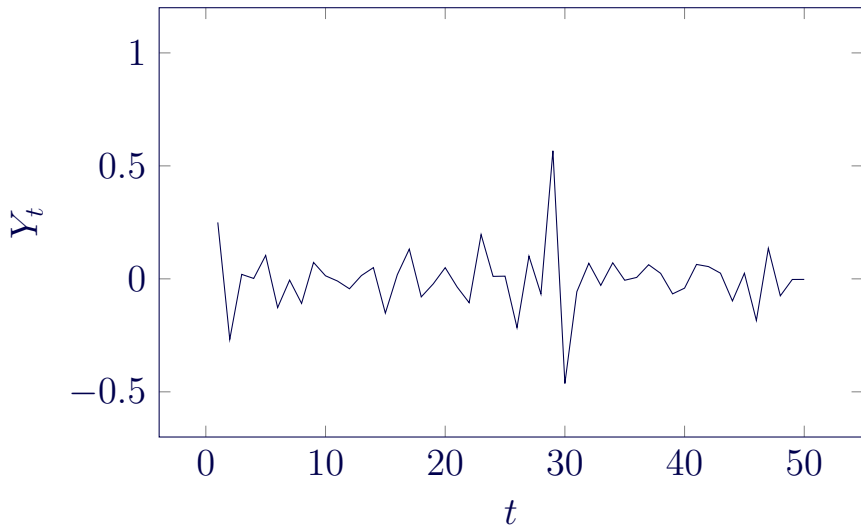
$$Y_t = \epsilon_t - 0.8\epsilon_{t-1} \quad (5)$$

### 3 Uncontaminated MA process



Source: Richards [2024, Figure 3].

# 3 IO — Innovation Outlier



Source: Richards [2024, Figure 3].

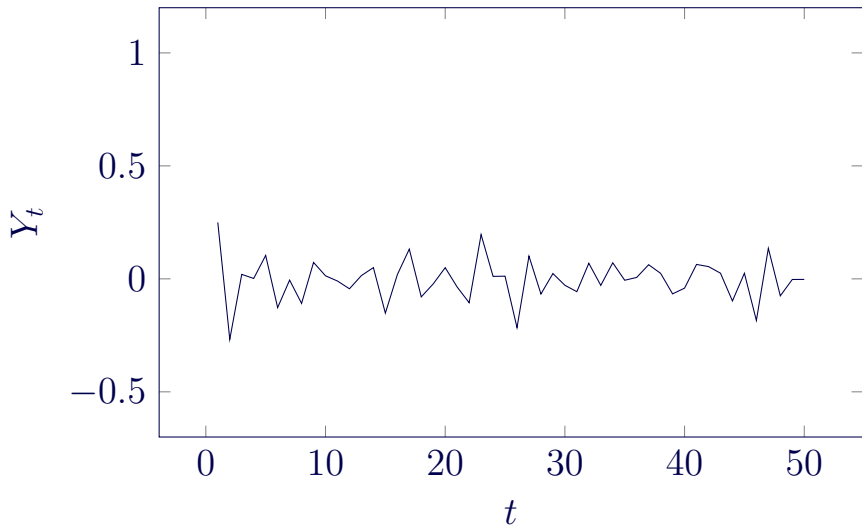
A modest outlier that is nevertheless integrated into the process.

## Example

A year with heavy winter mortality due to influenza, possibly with lighter mortality the following year.

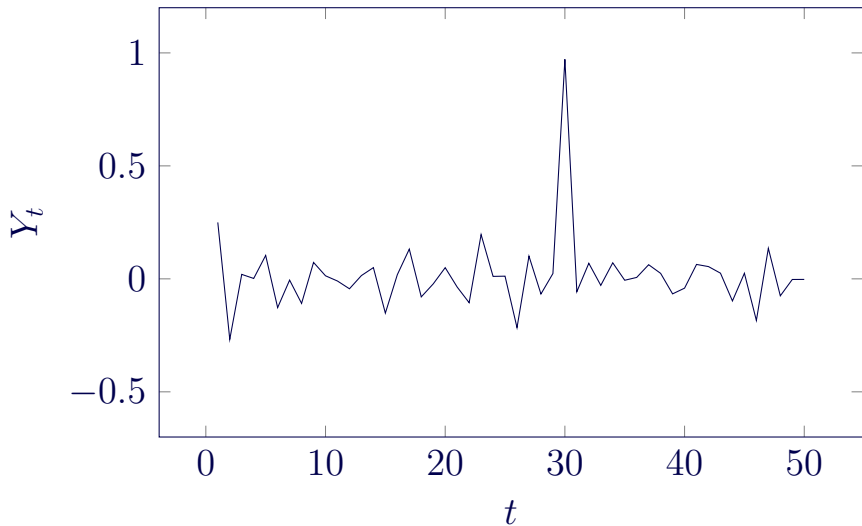
Handling: leave alone.

### 3 Uncontaminated MA process



Source: Richards [2024, Figure 3].

# 3 AO — Additive Outlier



Source: Richards [2024, Figure 3].

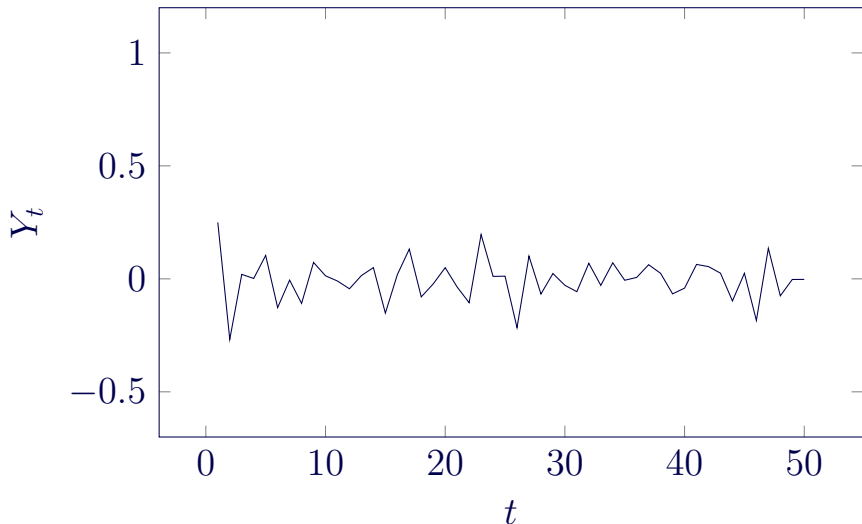
A more extreme outlier that is not integrated into the process.

## Example

War or pandemic in a single year.

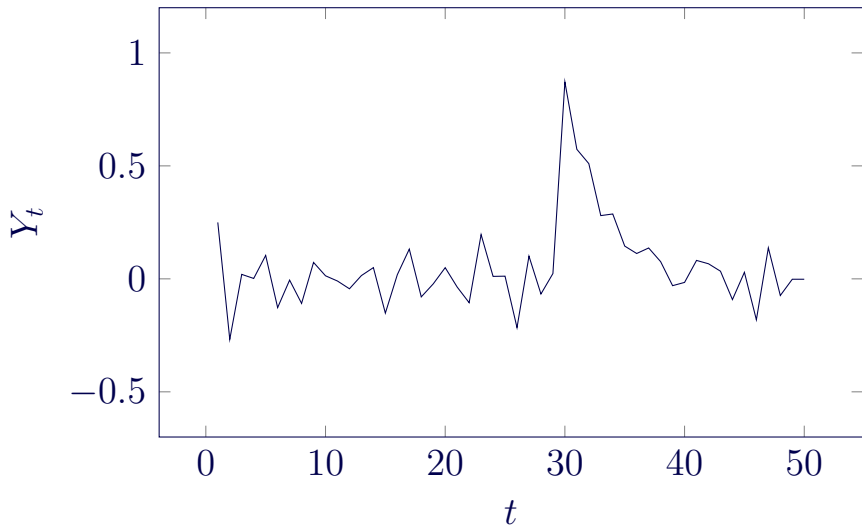
Handling: co-estimate the outlier effect to remove bias in other parameters.

### 3 Uncontaminated MA process



Source: Richards [2024, Figure 3].

# 3 TC — Temporary Change



Source: Richards [2024, Figure 3].

# 3 Temporary change



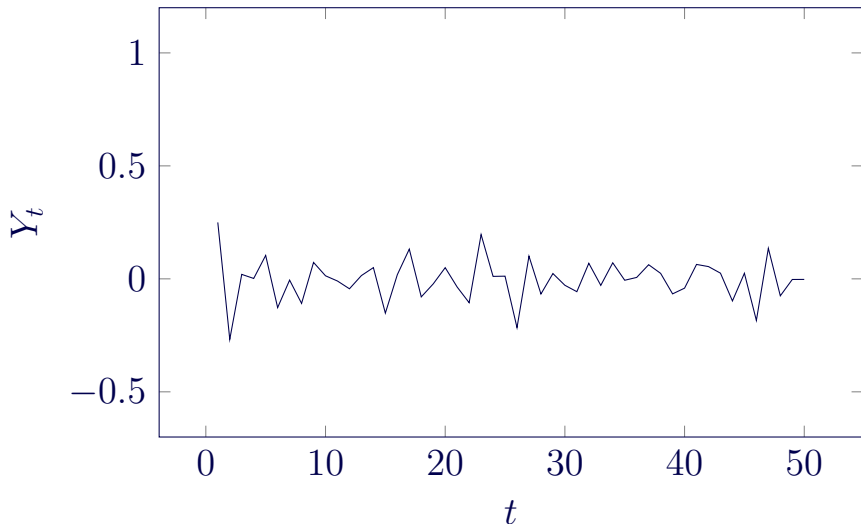
Two or more consecutive outliers that are not integrated into the process.

## Example

War or pandemic spread over more than one year.

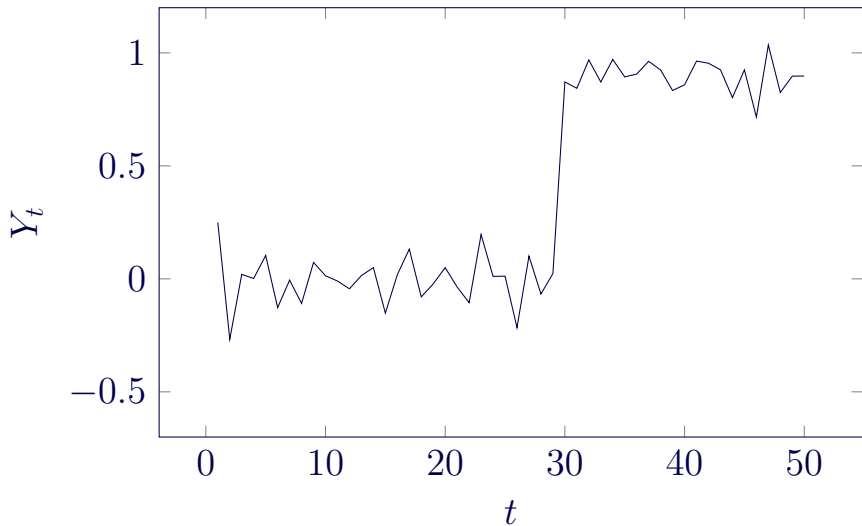
Handling: co-estimate the outlier effects to remove bias in other parameters.

### 3 Uncontaminated MA process



Source: Richards [2024, Figure 3].

# 3 LS — Level Shift



Source: Richards [2024, Figure 3].

Permanent change in level of process.

## Example

After German reunification in 1990, old-age mortality in East converged rapidly on levels in the West [Grigoriev et al., 2021].

Handling: review model or data period.

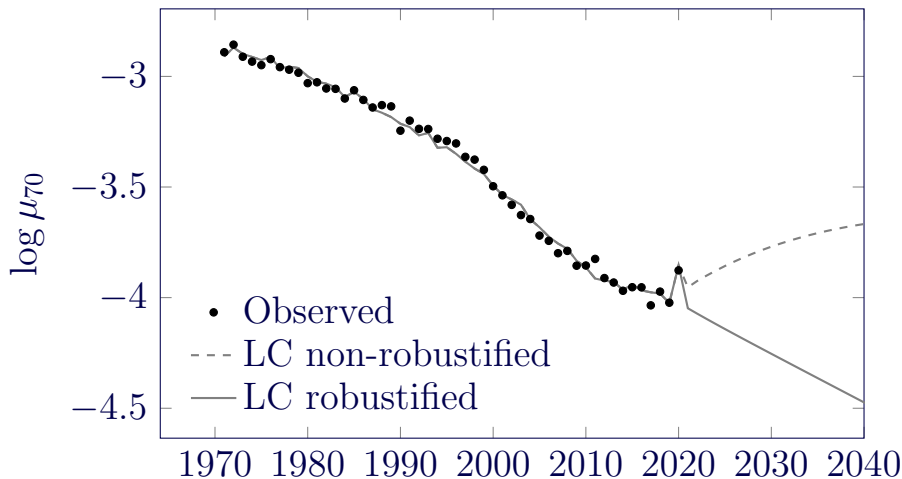
To robustify an ARIMA model, Chen and Liu [1993]:

1. Proposed test statistics to identify outliers.
2. Proposed further test statistics to *classify* outliers.

## Note

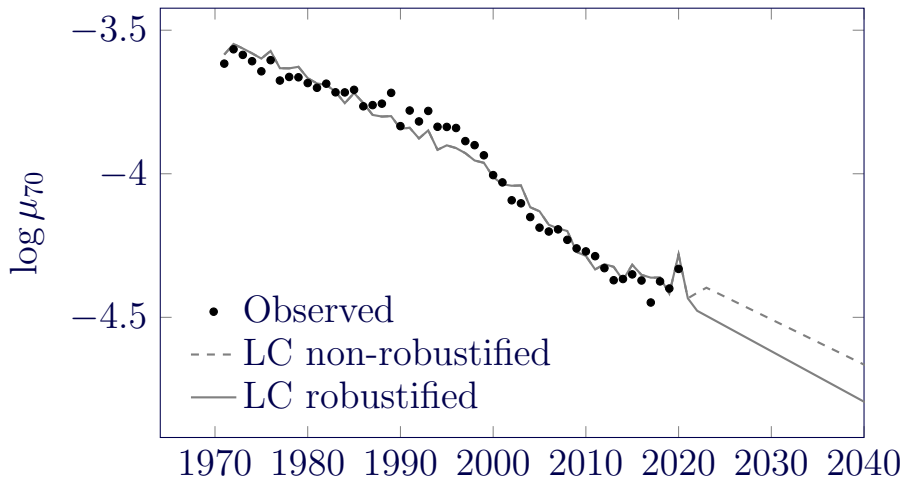
An outlier can be detected anywhere, but “it is impossible to empirically distinguish the type of an outlier at the very end of a series” [Chen and Liu, 1993, page 286].

# 3 Lee-Carter forecast — males



Source: Richards [2024, Figure 4(b)].

### 3 Lee-Carter forecast — females



Source: Richards [2024, Figure 4(a)].

Objective procedure allows for:

1. Identification of outlier location,
2. Classification of outlier type,
3. Estimation of outlier effect,
4. Calculation of “clean” starting point, and
5. Unbiased and undistorted forecast parameters.

# 4 Multivariate forecasts

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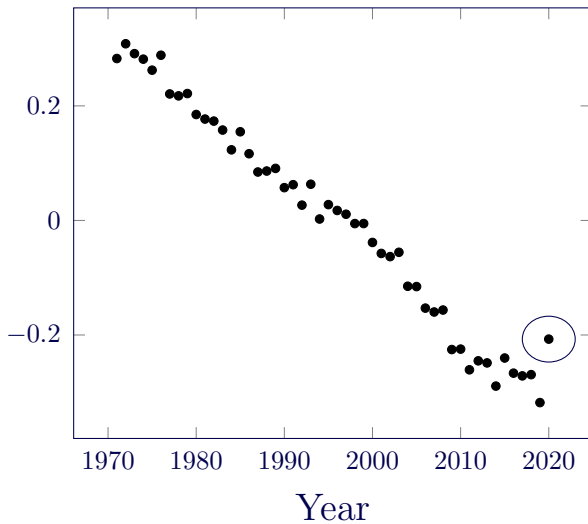


- A multivariate model has two or more time indices.
- Example from Cairns et al. [2006]:

$$\log \mu_{x,y} = \kappa_{0,y} + (x - \bar{x})\kappa_{1,y}$$

- $\hat{\kappa}_{0,y}$  and  $\hat{\kappa}_{1,y}$  are forecast jointly as a bivariate random walk with drift.

# 4 M9 example, $\hat{\kappa}_{0,y}$



Source: Richards [2024, Figure 6, left panel].

Various ways to identify outliers in multivariate data:

- Mahalanobis distance.
- Hadi [1992], Hadi [1994].
- Galeano et al. [2006].

- CBD models project  $\kappa$  using a  $p$ -dimensional random walk with drift.
- Calculate first difference  $\mathbf{z}_j = \Delta \kappa_j$ .
- Then  $\mathbf{z}_j$  has a multivariate normal distribution with mean vector  $\boldsymbol{\mu}$  and covariance matrix  $\boldsymbol{\Sigma}$ .

$$D_j = \sqrt{(z_j - \mu)^T \Sigma^{-1} (z_j - \mu)} \quad (14)$$

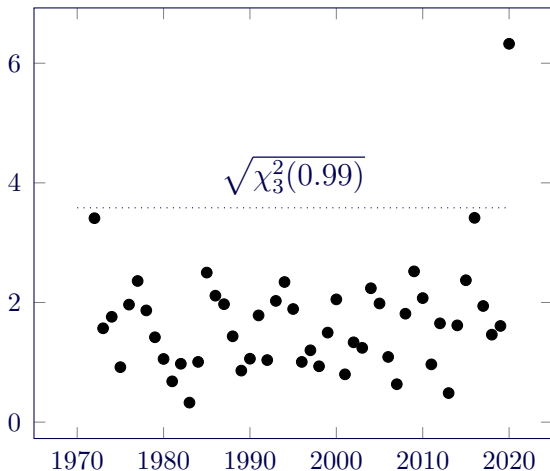
$$D_j^2 \sim \chi_p^2$$

.

# 4 Mahalanobis distance



Mahalanobis distance for  $\Delta\hat{\kappa}$  for M9.



Source: Richards [2024, Figure 8].

“Mahalanobis distance is not robust, as it is affected by masking and swamping” Hadi et al. [2009]

## Masking

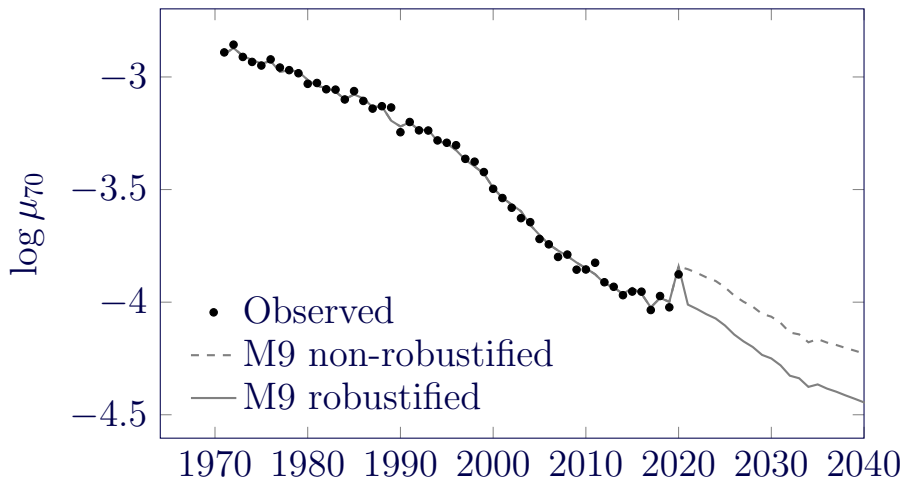
An outlier is hidden because it inflates the estimate of variance used to detect outliers.

## Swamping

Non-outliers have a large Mahalanobis distance because an outlier has distorted the mean of the process.

- Galeano et al. [2006].
- Use projection pursuit.
- See Richards [2024, Appendix C] for details of application to multivariate mortality indices using R.

# 4 Robust M9 forecast



Source: Richards [2024, Figure 9(b)].

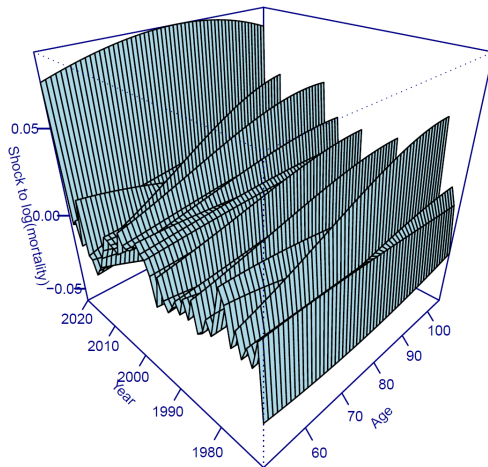
# 5 2D $P$ -spline model

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- Introduced by Currie et al. [2004].
- Extended by Kirkby and Currie [2010] to estimate period shocks. . .

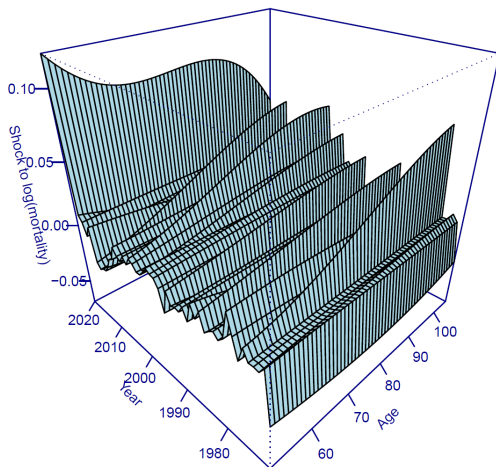
# 5 Period shocks (females)



Source: Richards [2024, Figure 10].

- Figure 10 identifies every period effect.
- Kirkby and Currie [2010] implemented scaling to dampen minor period effects.
- Figure 12 shows period effects with optimised scaling.

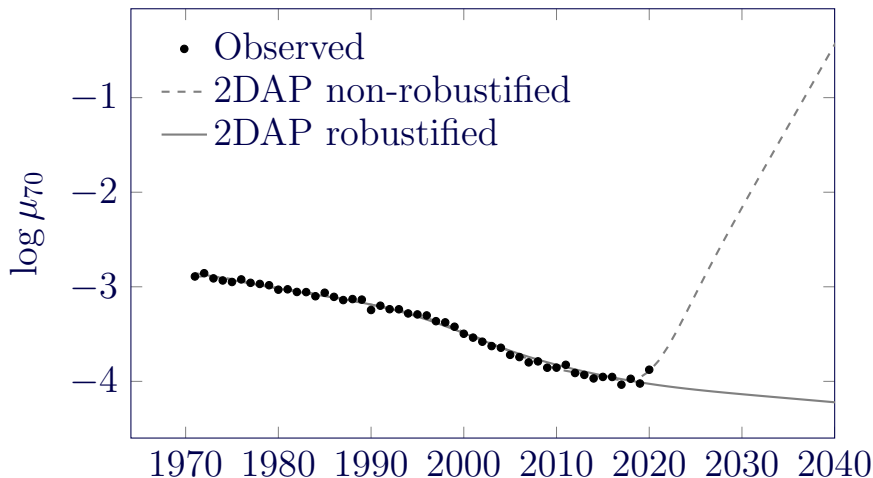
# 5 Scaled period shocks (females)



Source: Richards [2024, Figure 12].

- Forecast using *penalty function*.
- Estimating period shocks robustifies penalty forecast...

# 5 Robust 2DAP forecast



Source: Richards [2024, Figure 13(b)].

# 6 Conclusions

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Co-estimation of outliers and parameters:

1. Reduces bias in forecasting parameters.
2. Yields better starting points for forecasts.
3. Reduces variance in capital requirements.

## Univariate forecasting

- Lee-Carter and APC models.
- Use approach of Chen and Liu [1993].

## Multivariate forecasting

- Cairns-Blake-Dowd & Tang-Li-Tickle models.
- Use approach of Galeano et al. [2006].

## 2D $P$ -spline

- Currie-Durban-Eilers model.
- Use approach of Kirkby and Currie [2010].

- A. J. G. Cairns, D. Blake, and K. Dowd. A two-factor model for stochastic mortality with parameter uncertainty: theory and calibration. *Journal of Risk and Insurance*, 73:687–718, 2006. doi: 10.1111/j.1539-6975.2006.00195.x.
- C. Chen and L-M. Liu. Joint estimation of model parameters and outlier effects in time series. *Journal of the American Statistical Association*, 88(421): 284–297, 1993.

- I. D. Currie, M. Durban, and P. H. C. Eilers. Smoothing and forecasting mortality rates. *Statistical Modelling*, 4:279–298, 2004. doi: 10.1191/1471082X04st080oa.
- P. Galeano, D. Peña, and R. S. Tsay. Outlier detection in multivariate time series by projection pursuit. *Journal of the American Statistical Association*, 101 (474):654–669, 2006. doi: 10.1198/01621450000001131.

- P. Grigoriev, M. Pechholdová, M. Mühlichen, R. D. Scholz, and S. Klüsener. 30 Jahre Deutsche Einheit: Errungenschaften und verbliebene Unterschiede in der Mortalitätsentwicklung nach Alter und Todesursachen. *Bundesgesundheitsblatt*, pages 481–490, 2021. doi: 10.1007/s00103-021-03299-9.
- A. S. Hadi. Identifying multiple outliers in multivariate data. *Journal of the Royal Statistical Society, Series B (Methodological)*, 54(3):761–771, 1992. ISSN 00359246. URL <http://www.jstor.org/stable/2345856>.

- A. S. Hadi. A modification of a method for the detection of outliers in multivariate samples. *Journal of the Royal Statistical Society, Series B (Methodological)*, 56(2):393–396, 1994. ISSN 00359246. URL <http://www.jstor.org/stable/2345910>.
- A. S. Hadi, A. H. M. Rahmatullah Imon, and M. Werner. Detection of outliers. *WIREs Computational Statistics*, 1(1):57–70, 2009. ISSN 1939-0068. doi: 10.1002/wics.6.

- J. G. Kirkby and I. D. Currie. Smooth models of mortality with period shocks. *Statistical Modelling*, 10(2):177–196, 2010. doi: 10.1017/1471082X0801000204.
- R. D. Lee and L. Carter. Modeling and forecasting US mortality. *Journal of the American Statistical Association*, 87:659–671, 1992. ISSN 01621459. URL <http://www.jstor.org/stable/2290201>.

R. D. Martin, A. Samarov, and V. J. Vandaele. Robust methods for ARIMA models. In E. Zellner, editor, *Applied Time Series Analysis of Economic Data*, pages 153–177. Washington Bureau of the Census, 1983.

S. J. Richards. Robust mortality forecasting in the presence of outliers. *British Actuarial Journal (to appear)*, 2024.

Coronavirus graphic 🦠 from CDC

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