THE ACTUARIAL ORIGINS OF SURVIVAL MODELS

By S. J. Richards

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Richards, Stephen J.*

May 27, 2026

Abstract

Survival models based on individual lifetimes are a cornerstone of modern medical statistics. The foundations of survival analysis were laid by actuaries, driven by practical as well as theoretical benefits. However, technological limitations led the actuarial profession to leave the development of survival modelling to statisticians many decades later. This paper seeks to remind actuaries of their early leading role in this field, and perhaps to rekindle interest in what their forebears started, especially since computing resource has long ceased to be a limiting factor. This paper may also be of interest to non-actuaries in understanding the unique characteristics of actuarial data sets and of actuarial modelling requirements.

Keywords: survival models, mortality hazard, left-truncation, deduplication.

1 Introduction

This paper looks at the early development of survival models by actuaries publishing in English and German. We will see that actuaries developed many of the basic components of survival models, but that technology often imposed severe constraints on what could be achieved. These technological limitations may have played a role in actuaries leaving the field to be later developed by mathematicians and statisticians.

The plan of the rest of this paper is as follows. In Section 2 we state assumptions and define some terminology. Section 3 describes some defining features of actuarial data, including aspects not often encountered by statisticians. Section 4 looks at the landmark paper by Gompertz [1825]. Section 5 considers the advances made by Makeham [1860], which go further than merely adding a constant term to Gompertz's model. Section 6 looks at one of the earliest descriptions of life-office data preparation, which demonstrates a defining feature of actuarial data sets. Section 7 considers modelling by means of grouped counts or individual lifetimes. Section 8 briefly looks at the emergence of a modern expression for the probability of survival, while Section 9 looks at the role of early actuarial textbooks. Section 10 considers an early application of Makeham's model to the mortality experience of a pension scheme in the late 19th century. Section 11 looks at the actuarial origins of the Kaplan-Meier estimate, one of the most-used statistical tools in the modern world. Section 12 discusses why actuaries might have stopped developing this area of theory. Section 13 concludes.

2 Terminology

We define ${}_tp_x$ as the probability that a life aged x survives beyond age $x+t$:

$$ {}_tp_x = \Pr\left[\text{survives beyond age } x+t | \text{alive aged } x\right] \tag{1} $$

*stephen@longevitas.co.uk, Longevitas Ltd, 24a Ainslie Place, Edinburgh, EH3 6AJ, UK. http://orcid.org/0000-0001-6859-6347

where x and t are positive real numbers measuring years. Note that there is no requirement for x and t to be integers. Similarly, ${}_tq_x = 1 - {}_tp_x$ is the complementary probability of not surviving[1]. Shorthand expressions are $p_x = {}_1p_x$ and $q_x = {}_1q_x$.
The *instantaneous mortality hazard* at age x, μ_x, is defined as:

$$\mu_x = \lim_{t \to 0^+} = \frac{{}_tq_x}{t} \tag{2}$$

An alternative view of equation (2) is as a probability of death in a small interval of time, t:

$${}_tq_x = t\mu_x + o(t) \tag{3}$$

where $\lim_{t \to 0^+} \frac{o(t)}{t} = 0$. Equation (3) forms the basis of viewing mortality as a continuous series of "infinitesimal Bernoulli trials" [Macdonald and Richards, 2025, p.22]. Early actuaries used a variety of alternative names for μ_x, but we will use the term mortality hazard as it is standard in statistical literature; see Collett [2003, p.11] or Andersen et al. [1993, p.49].
The fundamental relationship linking ${}_tp_x$ and μ_x is [Dickson et al., 2019, eq.2.11]:

$${}_tp_x = e^{-\int_0^t \mu_{s+s} ds} \tag{4}$$

where $H_x(t) = \int_0^t \mu_{x+s} ds$ is the *integrated hazard* [Collett, 2003, p.12]. Alternatively, equation (4) can be expressed as:

$$H_x(t) = -\log {}_tp_x \tag{5}$$

where log denotes the natural logarithm. As we will see, early actuaries made extensive use of equation (5). Equation (4) shows that knowledge of the mortality hazard, μ_x, gives complete knowledge of every probability of survival or death, i.e. the mortality hazard is the most fundamental quantity for mortality analysis [Macdonald and Richards, 2025]. A *survival model* is a model for the mortality hazard, and estimation can be based on either of equations (4) or (5).

A major difference between actuarial users of survival models and other practitioners is the need to estimate the full hazard. Other practitioners are often only asking the question "does Group A have different mortality from Group B?". This often leads to use of the proportional hazards model [Cox, 1972], where the baseline (or reference) hazard is not estimated because it is not needed to answer the question. In contrast, the actuary asks "What are the hazards of Groups A and B?", as there is a fundamental requirement to estimate the baseline hazard along with all covariates to price cashflows contingent on survival or death.

3 Data

> *The records of a life assurance office afford a very important source from which to derive the materials for the construction of a mortality table*
>
> King [1887, p.7]

[1] Modern readers will note that this section is written in the style of an "actuary of the first kind" [Bühlmann, 1989, p.5], which is appropriate for an article discussing actuarial publications of the nineteenth century. An "actuary of the second kind" would instead start with the observation that the future lifetime of a life aged x is a random variable with distribution function ${}_tq_x$ and probability density function ${}_tp_x\mu_{x+t}$.

> *investigations are not made for academic reasons, but for business purposes*
>
> Hunter [1909-1910, p.263]

The fundamental unit of observation in a mortality study is an individual life [Macdonald and Richards, 2025]. An individual life i enters observation at age x_i years and is observed for a further t_i years. The indicator variable d_i signifies if the event of interest has occurred at age $x_i + t_i$, i.e. $d_i = 1$ in the case of death and $d_i = 0$ otherwise. $d_i = 0$ signifies that the observation is *right-censored*.

Lives enter observation at adult ages in actuarial work, and typically with $x > 50$ for pensions in payment and annuities. This means that observations are also *left-truncated*. Left-truncation is less common for non-actuarial survival analysis: Kaplan and Meier [1958, p.463] make only brief mention of left-truncation, while the term does not even appear in the index of Collett [2003]. However, more advanced texts do cover left-truncation [Andersen et al., 1993].

Lives also tend to enter an actuarial portfolio on a near-continuous basis, either through new business or through contingent benefits set up after death of the first life; see Richards and Macdonald [2025, Figures 3(a) and 4(a)] for examples. This continual recruitment to the portfolio is known as *staggered entry* among statisticians [Andersen et al., 1993]. However, actuaries sometimes need to distinguish between lives entering observation voluntarily (say because they take out a new policy) and lives entering involuntarily (say because their spouse died).

However, another feature of actuarial data sets is that administration records are orientated around policies or benefits, not lives. It is therefore relatively common for an individual to have two or more records in an actuarial data set; see Downes [1862] and Richards and Currie [2009]. Since statistical modelling requires the assumption of independent observations, actuaries have to perform a process of *deduplication* [Macdonald et al., 2018, Section 2.5]. Such a data-preparation step is seldom needed for other users of survival models, as records are typically already patient-orientated.

Left-truncation and deduplication are defining features of actuarial survival data. Actuarial data sets also typically have a larger scale, with modern portfolios sometimes covering hundreds of thousands or millions of lives.

4 Gompertz

Gompertz [1825] proposed the following functional form for the mortality hazard:

$$\mu_x = e^{\alpha + \beta x} \tag{6}$$

where α and β are real-valued parameters. Equation (6) allows only for age-related increasing mortality when $\beta > 0$, although Gompertz himself was clearly aware of a constant component in practice [Gompertz, 1825, Art.4, p.517]. The integrated hazard function for equation (6) is:

$$H_x(t) = \int_0^t \mu_{x+s} ds = \frac{(e^{\beta t} - 1)}{\beta} e^{\alpha + \beta x}, \qquad \beta \neq 0 \tag{7}$$

Equation (6) is the modern expression of Gompertz's law, which is preferred because it obviates the requirement to impose constraints when estimating parameters. Gompertz's original definition used $\mu_x = aq^x$, where a and q are positive real numbers; the q here has nothing to do with the q_x notation in Section 2, which is another reason to prefer the modern notation of equation (6). To equate the two definitions we use:

$$\begin{aligned} \alpha &= \log a \\ \beta &= \log q \end{aligned} \tag{8}$$

As equation (6) suggests, Gompertz's 1825 law was expressed in continuous time, as is evident from his reference to "infinitely smaIl intervals of time" (page 518). He referred to his measure as the "intensity of [...] mortality" (page 518), a term still occasionally used today as a synonym for the mortality hazard[2].

Gompertz's paper addressed many practical actuarial problems of his day, including:

- Interpolating mortality rates at any age. At a time when mortality data were often published in ten-year intervals of age, there was a pressing commercial need for actuaries to calculate mortality rates at any age. Gompertz's model could be fitted to any mortality data, and mortality rates thus derived for any intervening age. Indeed, although it wasn't regarded as significant at the time, Gompertz's model also allowed extrapolation of mortality rates to ages beyond the data.
- A closed-form expression for the survival probability between any two ages. This reduced the amount of calculation required and thus the scope for error. As we will see in Section 12, most actuarial calculations of the time were done manually, so these were important practical benefits.
- A closed-form expression for the joint survival probability. Calculating survival probabilities for the first failure of joint lives is simpler when you can just add hazard rates. In 19th Century Britain, annuities contingent on multiple lives were not uncommon.

Gompertz tabulated survival probabilities, not annual mortality rates. With his modelling of the mortality hazard and his focus on survival probabilities, Gompertz [1825] was perhaps the earliest advocate of survival models for actuarial work.

5 Makeham

Makeham [1860] extended Gompertz's mortality law to include a constant term:

$$\mu_x = e^{\epsilon} + e^{\alpha+\beta x} \tag{9}$$

where ϵ is a new real-valued parameter like α, β. As with equation (6), equation (9) is the modern expression of Makeham's law, which avoids having to impose constraints on the parameters during estimation. The integrated hazard function for equation (9) is:

$$H_x(t) = \int_0^t \mu_{x+s} ds = te^{\epsilon} + \frac{(e^{\beta t} - 1)}{\beta} e^{\alpha+\beta x}, \qquad \beta \neq 0 \tag{10}$$

Makeham expressed his law differently to equation (9), but his 1860 paper is interesting because the parameters are estimated from a system of three consecutive twenty-year survival probabilities (p.304):

$$\begin{aligned} \log {}_{20}p_{20} &= a = -H_{20}(20) \\ \log {}_{20}p_{40} &= b = -H_{40}(20) \\ \log {}_{20}p_{60} &= c = -H_{60}(20) \end{aligned} \tag{11}$$

[2] Lambert [1772, p.510 and Figure I] used a geometric argument to define *Lebenskraft*, which was equal to the inverse of the mortality hazard. However, Lambert used this as a descriptive feature of the survival curve, rather than the basis of calculation, and Lambert's mortality rates are all based on single years of age with no mention of using smaller intervals.

where we have substituted p for Makeham's original π notation for the survival probability and included the integrated hazards for clarity. The equation system in (11) can then be expressed in terms of the parameterisation of equation (9) as follows:

$$\begin{aligned} e^{20\beta} &= \frac{b-c}{a-b} \\ e^{\alpha} &= \frac{\beta(a-b)^4}{(b-c)(a+c-2b)^2} \\ 20e^{\epsilon} &= \frac{b^2-ac}{a+c-2b} \end{aligned} \tag{12}$$

The latter term in equation system (12) is referred to as "a certain uniform quantity" in Makeham [1860, p.303]. Having estimated the three parameters, Makeham could then deduce estimates of mortality rates at any intervening age (page 306). Makeham [1866, p.315] later published an explicit formula for his mortality law and a closed-form expression for the survival function.

Makeham [1867] presented more detail on the same model. Like Gompertz before him, Makeham knew the advantages of working in continuous time over discrete time. He began his 1867 paper by calling the hazard "indispensable" to work in mortality. Makeham himself used the term "force of mortality", which was used even earlier by Edmonds [1832, Chapter I]; this term is still used by many Anglophone actuaries today. Makeham's argument was that an interval-based mortality rate is an imperfect measure when mortality levels are high, as it doesn't account for the changing (reducing) number of lives exposed to risk during the interval. Makeham's solution to this problem was to shrink the age interval to zero ("diminish Δx without limit") and thus work in continuous time. Like Gompertz, Makeham used the continuous-time mortality hazard and survival probabilities, i.e. survival models.

Makeham's law in equation (9) adds a constant term to Gompertz's law, and Makeham demonstrated the value of this in his 1867 paper using five different example data sets. However, the importance of Makeham [1867] went much further than merely adding a single parameter to Gompertz's model:

- He showed how competing decrements can be handled in continuous time (pages 329-330), and
- He further showed that continuous-time methods work when there is immigration (new entrants) and emigration (withdrawals) (pages 332-333).

Richards and Macdonald [2025] give a detailed treatment of the practical benefits for modern actuaries of working in continuous time.

At the time Makeham wrote his papers there were advocates for using raw data, rather than fitting mortality models. To such critics Makeham replied:

> *the very worst course that could possibly be adopted is to pin our faith upon the crude results of observation [...] and hoodwink ourselves into the belief that in so doing we are following the path indicated by experience.*
>
> Makeham [1867, p.346]

Makeham [1867] was therefore also an early advocate of using parametric models to smooth out random variation.

6 Downes

Downes [1857] published a mortality analysis of insured lives and then followed with an account of the data processing in Downes [1862]. Lidstone [1946] provides a historical review of this data-processing system, which was based on paper cards.

Downes [1862] is interesting because it focuses on the detailed collection of *individual* exposure data, regardless of how those data might be later analysed. Downes was also careful to collect fractional years of exposure without truncation or rounding:

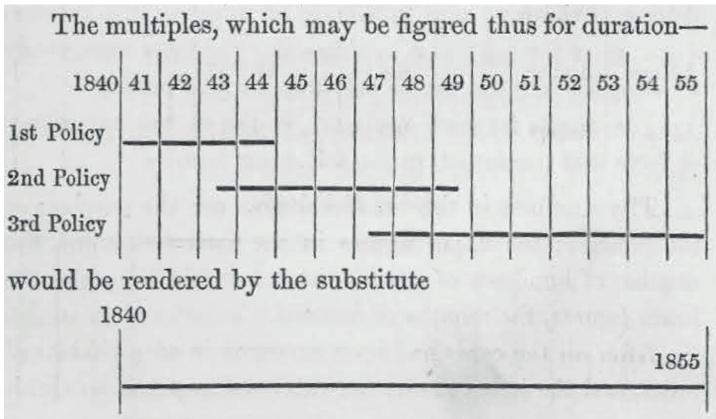
> *the exact number of years of age passed through, as well as the fraction in days at the beginning and termination of each policy, is given.*
>
> Downes [1862, p.4]

Thus, the data collected and processed by Downes would have been suitable for a survival model calibrated using individual lifetimes if the mathematics and technology been available (see Section 12).

A key part of this processing lies in taking dates and calculating ages and policy durations in years. Downes [1862, p.7] carried this out through a process of decimalising each day in a year. Downes standarised on a 365-day non-leap year, although an allowance for leap years can also be made [Macdonald et al., 2018, p.38].

A defining feature of actuarial data is that it is policy- or benefit-orientated, these being the units of contractual liability for a life office. It is therefore possible for a single individual to have multiple policies or benefits. Downes [1862, p.14] described an early instance of deduplication, which he called "substitutes for multiples". His example, shown in Figure 1, exhibits three policy records for the same individual with left truncation and right censoring at various ages and the corresponding replacement record for the individual life.

Figure 1: Substitution of three overlapping policy exposures with a single lifetime exposure record in Downes [1862, p.14].



Central to deduplication is the idea of a *deduplication key*, i.e. a combination of data fields that together reliably identifies a unique individual [Macdonald et al., 2018, p.28]. Downes [1862, pp.12-13] used a combination of surname, forename and date of birth, which was adequate for his portfolio of 11,945 policies. Once a deduplication key has been selected, the data need to be sorted, and for Downes this was a decidedly manual task:

> *a process which would afford pleasant fireside amusement to any domesticated actuary and his family.*
>
> Downes [1862, p.18]

However, modern portfolios run to hundreds of thousands of policy records. Such portfolios require more complex deduplication keys to avoid false matches [Macdonald et al., 2018, Section 2.5] and an electronic computer to do the sorting.

Downes (p.19) identified an average of 1.28 assurance policies per person. In a similar deduplication exercise for pension annuities, Richards and Currie [2009, Table 1] found an average of 1.24 policies per person, with a strong correlation between the number of policies per person and socio-economic profile. Deduplication has always been an essential business process to gain insight into the nature of actuarial liabilities.

7 Grouped counts versus individual lives

Downes [1862] showed that insurers have always held details of the individual lives assured, and that it has always been necessary to process individual records to identify duplicates. A natural extension of this would be to model the mortality of individual lives, these being the fundamental unit of mortality work [Macdonald and Richards, 2025].

However, computational technology was a major constraint (see Section 12). Early actuaries therefore had to make simplifying assumptions, such as that of a constant hazard for a year of age. This allowed the summing of fractional exposure times in a given age interval and the summing of the corresponding deaths; see Macdonald and Richards [2025, Section 4.7]. This can still be regarded as a survival model as it is based on the mortality hazard, and the error introduced by the assumption of the constant hazard can be minimised by keeping the age intervals as small as possible.

This grouped-counts approach was the only practical option in the nineteenth century due to the limits of computational technology. It remains a useful simplification even with present-day technology. However, the grouped-counts approach imposes restrictions, such as the ability to treat continuous individual characteristics as continuous variables (for example, pension size, BMI or blood pressure).

8 Lazarus

Both Gompertz [1825] and Makeham [1860] used l_x to denote the number of lives aged x, and both used the progression in $\log l_x$ to estimate the parameters of their mortality laws[3]. Like Gompertz [1825] and Makeham [1860] before him, Lazarus [1862] presented his arguments in continuous time. He referred to "the probability to die in the next moment"(p.285), which is the rationale for equation (3). With a slight change in arrangement, Lazarus [1862, p.285] further noted that the expected number living, l_x, was given by:

$$l_x = l_0.e^{-\phi(x)} \tag{13}$$

where ϕ is the integrated hazard function since birth, i.e. $H_0(x)$. Lazarus [1862] thus anticipated the modern expression of the survival probability in equation (4).

[3]A modern actuary would describe l_x as the "expected number of survivors at age x" [Seal, 1977, p.429].

9 King

Makeham [1860] used equation system (11) to estimate the parameters of his mortality law. In fitting a Makeham law King [1887, p.81] wanted to "make use of as large a portion as possible of [the] data". This required many more calculation steps, which were made possible because of technology unavailable to earlier authors (see Section 12).

King [1887, p.73] uses a parameterisation of Makeham's law that we can express in terms of the parameters in equation (9) as follows:

$$\begin{aligned} \alpha &= \log(-\log c \times \log g) \\ \beta &= \log c \\ \epsilon &= \log(-\log s) \end{aligned} \tag{14}$$

King [1887, p.81] created systems of equations using the closed-form expression of the Makeham survival function. The essence of King's methodology can be summarised by first noting the following summations for $\beta \neq 0$:

$$\sum_{i=0}^{t-1} \log {}_tp_{x+i} = \sum_{i=0}^{t-1} -H_{x+i}(t) \quad = -t^2 e^{\epsilon} + \frac{(e^{\beta t}-1)^2}{\beta(1-e^{\beta})} e^{\alpha+\beta x} \tag{15}$$

$$\sum_{i=0}^{t-1} \log {}_tp_{x+t+i} = \sum_{i=0}^{t-1} -H_{x+t+i}(t) = -t^2 e^{\epsilon} + \frac{(e^{\beta t}-1)^2}{\beta(1-e^{\beta})} e^{\alpha+\beta(x+t)} \tag{16}$$

$$\sum_{i=0}^{t-1} \log {}_tp_{x+2t+i} = \sum_{i=0}^{t-1} -H_{x+2t+i}(t) = -t^2 e^{\epsilon} + \frac{(e^{\beta t}-1)^2}{\beta(1-e^{\beta})} e^{\alpha+\beta(x+2t)} \tag{17}$$

from which we can see that King's method is a survival model because it is based on the integrated hazard function. Taking the first differences of equations (15)-(17) we get:

$$\Delta_1 = \sum_{i=0}^{t-1} \log {}_tp_{x+t+i} - \sum_{i=0}^{t-1} \log {}_tp_{x+i} \quad = \frac{(e^{\beta t}-1)^3}{\beta(1-e^{\beta})} e^{\alpha+\beta x} \tag{18}$$

$$\Delta_2 = \sum_{i=0}^{t-1} \log {}_tp_{x+2t+i} - \sum_{i=0}^{t-1} \log {}_tp_{x+t+i} = \frac{(e^{\beta t}-1)^3}{\beta(1-e^{\beta})} e^{\alpha+\beta(x+t)} \tag{19}$$

from which King obtained the following (p.82):

$$\frac{\Delta_2}{\Delta_1} = e^{\beta t} \tag{20}$$

which allows the estimation of β from the various empirical values of ${}_tp_x$, and hence the values of α and ϵ, as follows:

$$\begin{aligned} \hat{\beta} &= \frac{1}{t} \log\left(\frac{\hat{\Delta}_2}{\hat{\Delta}_1}\right) \\ \hat{\alpha} &= \log\left[\frac{\hat{\beta}(1-e^{\hat{\beta}})}{(e^{\hat{\beta}t}-1)^3}.\left(\hat{\Delta}_1\right)\right] - \hat{\beta}x \\ \hat{\epsilon} &= \log\left[\frac{(e^{\hat{\beta}t}-1)^2}{\hat{\beta}(1-e^{\hat{\beta}})}.e^{\hat{\alpha}+\hat{\beta}x} - \sum_{i=0}^{t-1} {}_t\hat{p}_{x+i}\right] - 2\log t \end{aligned} \tag{21}$$

where $\hat{\Delta}_1$ and $\hat{\Delta}_2$ denote expressions calculated using ${}_t\hat{p}_x$, the observed survival rate from age x to $x+t$ based on the data.

10 Karup

Karup [1893] provided a demonstration of King's method applied to a pension scheme open to new entrants and with various modes of exit besides death. Karup used census formulae to estimate ${}_t\hat{p}_x$ — see the details in Appendix A. The parameterisation in Karup [1893, p.38] is similar to that of King [1887, p.79] and we can express Karup's parameters g, q and s in terms of equation (9) as follows:

$$\begin{aligned} \alpha &= \log(-\log q \times \log g) \\ \beta &= \log q \\ \epsilon &= \log(-\log s) \end{aligned} \tag{22}$$

Note that the log function in equation (22) is the natural logarithm, whereas the logarithms in Karup [1893, pp.37-38] use base 10. A comparison of the parameter estimates using Karup's approach and maximum-likelihood is given in Table 1. The closeness of the two fitted hazards is evident in Figure 2. The modern maximum-likelihood fit would be regarded as better because it has an implicit weighting by the volume of data at each age. However, Karup's hazard differs by at most 4.2%, and this "error" could be regarded as immaterial for the purposes of valuing the society's liabilities on an ongoing basis. The underlying data are reproduced in Appendix A.

Table 1: Comparison of parameter estimates using (a) Karup's approach using grouped counts, and (b) maximum-likelihood estimation using data at individual ages.

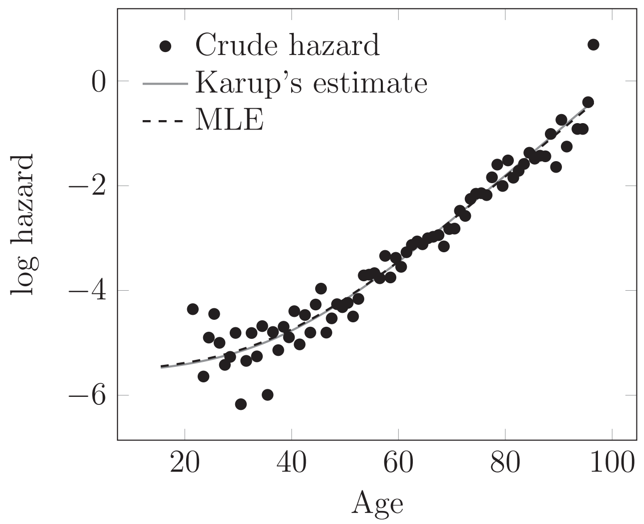
Methodology	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\epsilon}$
(a) Karup [1893, p.38]	-8.80804	0.087268	-5.62118
(b) Maximum likelihood	-8.73382	0.086071	-5.60040

Karup used data spanning 1850-1889 to obtain a large enough number of deaths. Karup's analysis is of period mortality, although there may well be material cohort effects in such a long investigation period.

11 Böhmer

Kaplan and Meier [1958] introduced a product-limit estimator for the survival curve. The Kaplan-Meier estimate is well suited to actuarial work on account of its ability to handle left-truncation

Figure 2: Plot of hazards for mortality-experience data in Karup [1893], together with the fitted Makeham hazards using (a) Karup's approach and (b) maximum likelihood (ML). The two estimated hazards are never more than 4.2% different across the age range and are barely distinguishable in the plot. This suggests that the fitting approaches of Gompertz and Makeham can be used to find good initial values; see Appendix B.



and right-censoring; see Richards and Macdonald [2025] for examples and R code to produce the Kaplan-Meier estimate for any given outset age.

The Kaplan-Meier estimate is a cumulative product, where the death times are sorted in increasing order and the cumulative product is taken of the ratio of lives in-force immediately before and after each death time. As Kaplan and Meier themselves noted, this idea was not new:

> *this estimate is a limiting case of the actuarial estimates. It was proposed as early as 1912 by Böhmer [...] but seems to have been lost sight of by later writers*
>
> Kaplan and Meier [1958, p.461]

The underlying methodology is the same, as can be seen in the comparison of the product-limit estimators in Figure 3.

Figure 3: Product-limit estimators from Böhmer [1912] and Kaplan and Meier [1958].

$1-\gamma_h = \Pi_h \dfrac{A_n}{A_{n-1}}$	Böhmer [1912, equation 4]
$\hat{P}(t) = \prod_{j=1}^{k} (n_j'/n_j)$	Kaplan and Meier [1958, equation 2b]

Böhmer was an academic actuary who also worked for a time at the German insurance regulator [DGVFM, 1957, p.134]. He was motivated by the challenge of estimating mortality rates in the presence of other movements besides death. He illustrated his proposal with a worked example with the following events [Böhmer, 1912, p.331]:

- deaths (the decrement of interest),
- disability (a competing decrement, treated as a censoring event),

- voluntary withdrawal (another competing decrement, treated as a censoring event), and
- new entrants (left-truncated new observations entering during the investigation period).

Böhmer's comprehensive example shows left-truncation, staggered entry and competing risks. (Böhmer didn't use those terms, of course, as they had yet to be invented.) There can be no doubt that the Kaplan-Meier estimator is suitable for actuarial work — it was first proposed by an actuary for a specifically actuarial problem. Nevertheless, it was Kaplan and Meier who made the intellectual leap to (i) estimate the entire survival curve, (ii) show that the product-limit estimate is also the maximum-likelihood estimate, and (iii) provide an approximate confidence band for the estimated survival curve (an improved expression for the confidence bands is documented in Kalbfleisch and Prentice [2002, p.17]).

12 Technology

The preceding sections show that actuaries laid the foundations for survival models long before anyone else. So why did actuaries not build on their earlier work? Part of the answer lies in the computational tools available:

> *Two different but related types of machinery came into use at the end of the 19th century to help with calculating tasks: mechanical calculators and tabulating machines.*
>
> Norberg [1990, p.760]

At the time of Gompertz [1825] and Makeham [1860] there was a pressing business need to calculate mortality rates for any age, not just the ten-year intervals of age used in tabulated mortality statistics. However, calculations also had to be economically efficient. For example, the preface to Gray [1849] thanks a number of individuals for each performing thousands of presumably manual calculations. By the 1860s the height of computational sophistication was the arithmometer, a mechanical desktop calculator illustrated in Figure 4.
Arithmometers first became commercially available in 1851 [Campbell-Kelly, 1992, p.126] and actuaries were early users:

> *One of the most prominent areas in which the arithmometer came into use in England was in insurance. Actuaries were probably the first professional group to adopt it and adapt their work to its capabilities.*
>
> Johnston [1997, p.18]

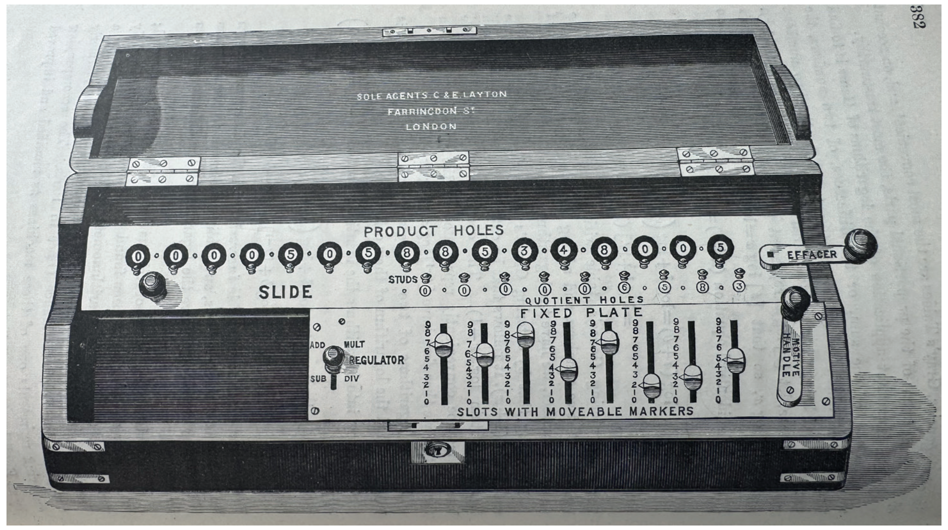
However, the first attempts to use the arithmometer were not always smooth:

> *initial experience with the arithmometer was not particularly good. Records within the Prudential show that the machines frequently broke down*
>
> Campbell-Kelly [1992, p.127]

Nevertheless, the arithmometer remained the pinnacle of actuarial calculation technology for decades:

> *Calculating machines of various kinds have been constructed; but the only one which has been found practically useful in the daily work of the actuary is the arithmometer.*
>
> King [1887, p.381]

Figure 4: The arithmometer was a mechanical device capable of addition, multiplication, subtraction and division, but not any of the more advanced functions like logarithms. Source: engraving from King [1887, p.382].



In this environment, survival models had much to offer. A parametric formula fitted to interval-grouped data could give mortality at any age, and closed-form expressions for the survival function allowed efficient calculation.

However, Böhmer's product-limit estimator was probably not developed further because of its complex requirements for sorting and selecting subsets; Seal [1954, p.149] described it as "excessively laborious to evaluate", which is true if one doesn't have an electronic computer. Although the estimators in Figure 3 are conceptually very simple, they require dynamic calculations and comparisons to establish the counts involved. However, in 1912 the height of sorting technology was the punched card, which could only sort by pre-determined categories [Yates, 1993, p.21] or policy number [Fackler, 1902, p.15]. Card-sorting machines used for mortality investigations around 1900 were not quick either:

> *so rapidly does the machine do its work, that the three million cards will be sorted within four months.*
>
> Fackler [1902, p.13]

Kaplan and Meier [1958] appeared 46 years after Böhmer [1912], and even then this was still too far ahead of the available technology:

> *The Kaplan-Meier paper was a sleeper hit, receiving almost no citations until computing power boomed in the 1970s, making the methods accessible to non-specialists.*
>
> van Noorden et al. [2014, p.552]

Mathematics can clearly run far ahead of the computational capabilities of the time. This may partly explain why Böhmer's proposal remained a near-forgotten curiosity, and why actuarial development of survival models stagnated for so long.

13 Conclusions

Beginning with Gompertz [1825], actuaries drove the development of models based on the mortality hazard. The reasons to prefer continuous-time modelling over discrete-time modelling have been recognised since Makeham [1860], and these reasons still apply today [Richards and Macdonald, 2025]. However, in the nineteenth century a key additional appeal of the mortality hazard was easier manual computation using closed-form expressions for the survival curve.

With Böhmer [1912] an actuary laid the foundation for one of the most successful statistical methodologies in use today, the Kaplan-Meier estimator [Kaplan and Meier, 1958]. However, commercially available technology was incapable of implementing it, so Böhmer's idea remained a mathematical curiosity for decades.

The most fundamental unit of mortality modelling is the individual life [Macdonald and Richards, 2025]. Downes [1862] showed that the administration records of life-insurance companies have always been suitable for this. However, technological limitations again meant that mortality historically had to be modelled as grouped counts, rather than individual lifetimes. These technological restrictions no longer apply, so actuaries can use the unique data resources at their disposal to model mortality in continuous time at the level of the individual life.

14 Acknowledgements

The author thanks Professor Angus S. Macdonald for comments on earlier drafts and bringing the preface of Gray [1849] to the author's attention. Thanks also to Professor Moshe A. Milevsky, Ross R. Ainslie, Patrick Kelliher and Stefan J. Ramonat for comments on later drafts. Any errors in the paper are the sole responsibility of the author. The author also thanks David Raymont, Ken Bogle and Greg McGhee at the Institute and Faculty of Actuaries for arranging the loan of King [1887] and Karup [1893] and for providing a scanned copy of Böhmer [1912]. This document was typeset in LaTeX. Graphs were created with tikz [Tantau, 2024] and pgfplots [Feuersänger, 2015].

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Appendices

A Appendix A

Tables 2 and 3 show the data for the main (male) lives from Karup [1893, p.63]. Karup's definitions of the column headings are as follows:

- x is the age as at 1st January of the year of entry/exit.
- B_x is the count of existing members already of given age at 1st January 1850.
- E_x is the count of "new entrants" to this age group, i.e. this covers both genuine new entrants and those existing members achieving this age.
- A_x is the count of exits not due to death at given age on 1st January of the exit year.
- T_x is the count of deaths at the given age on 1st January of year of death.
- D_x is the count of members at the given age as of 31st December 1889.
- F_x is the count of all lives either starting in the given age or achieving it during the observation period, i.e. $F_x = B_x + E_x$.
- G_x is the count of all lives exiting at the given age, or who could have done had they not died, i.e. $G_x = T_x + D_x$.
- S_x is the cumulative sum of F_x, i.e. $S_x = \sum_{j \leq x} F_j$.
- S'_x is the cumulative sum of G_x, i.e. $S'_x = \sum_{j \leq x} G_j$.
- $C_x = S_x - S'_{x-1}$.
- $R_x = C_x - (E_x + A_x)/2$.

Karup's definition of R_x is then an approximation of the effective number of lives for a binomial-style model for mortality. This approximation is described by Lidstone [1894, p.305], who quantifies the possible error.

For the maximum-likelihood model in Table 1 and Figure 2 we use the pseudo-Poisson model of Macdonald and Richards [2025], which happens to have a likelihood, L, that is proportional to the likelihood of the ordinary Poisson model as follows:

$$L \propto \prod_{x=15}^{96} e^{-\mu_x (R_x - \frac{1}{2}T_x)} \mu_x^{T_x} \tag{23}$$

where μ_x is the Makeham mortality hazard in equation (9) and $R_x - \frac{1}{2}T_x$ is an approximation of the central exposed-to-risk at age x.

Table 2: Data from Karup [1893, p.63] for ages 15–55 over the period 1850-1889. An electronic version of this data can be downloaded at https://www.longevitas.co.uk/information-matrix-page/johannes-karup

Age, x	B_x	C_x	A_x	T_x	D_x	F_x	G_x	S_x	S'_x	C_x	R_x
15		1				1		1		2	1.5
16		1				1		2		2	1.5
17								2		2	2
18	1	5				6		8		8	5.5
19		13				13		21		21	14.5
20		26	2			26	2	47	2	47	33
21		68	1	1	1	68	3	115	5	113	78.5
22	4	112	3		3	116	6	231	11	226	168.5
23	3	127	8	1	15	130	24	361	35	350	282.5
24	4	155	7	3	23	159	33	520	68	485	404
25	5	126	9	6	36	131	51	651	119	583	515.5
26	8	124	14	4	26	132	44	783	163	664	595
27	5	119	11	3	21	124	35	907	198	744	679
28	14	124	15	4	40	138	59	1045	257	847	777.5
29	22	118	13	7	25	140	45	1185	302	928	862.5
30	20	127	16	2	27	147	45	1332	347	1030	958.5
31	20	104	16	5	41	124	62	1456	409	1109	1049
32	30	83	15	9	44	113	68	1569	477	1160	1111
33	31	74	10	6	32	105	48	1674	525	1197	1155
34	25	50	16	11	32	75	59	1749	584	1224	1191
35	23	49	20	3	41	72	64	1821	648	1237	1202.5
36	21	47	13	10	46	68	69	1889	717	1241	1211
37	19	26	14	7	32	45	53	1934	770	1217	1197
38	29	35	9	11	43	64	63	1998	833	1228	1206
39	28	36	11	9	31	64	51	2062	884	1229	1205.5
40	29	41	8	15	41	70	64	2132	948	1248	1223.5
41	31	33	11	8	40	64	59	2196	1007	1248	1226
42	31	22	7	14	37	53	58	2249	1065	1242	1227.5
43	29	28	6	10	38	57	54	2306	1119	1241	1224
44	25	26	6	17	36	51	59	2357	1178	1238	1222
45	32	28	3	23	37	60	63	2417	1241	1239	1223.5
46	41	19	3	10	34	60	47	2477	1288	1236	1225
47	21	18	9	13	29	39	51	2516	1339	1228	1214.5
48	33	15	7	17	29	48	53	2564	1392	1225	1214
49	34	11	6	16	39	45	61	2609	1453	1217	1208.5
50	23	14	3	17	23	37	43	2646	1496	1193	1184.5
51	17	13	3	13	31	30	47	2676	1543	1180	1172
52	24	17	5	18	24	41	47	2717	1590	1174	1163
53	30	11	4	28	28	41	60	2758	1650	1168	1160.5
54	31	12	2	28	33	43	63	2801	1713	1151	1144
55	23	4	2	28	29	27	59	2828	1772	1115	1112

Table 3: Data from Karup [1893, p.63] for ages 56–96 over the period 1850-1889. An electronic version of this data can be downloaded at https://www.longevitas.co.uk/information-matrix-page/johannes-karup.

Age, x	B_x	C_x	A_x	T_x	D_x	F_x	G_x	S_x	S'_x	C_x	R_x
56	35	6	2	25	31	41	58	2869	1830	1097	1093
57	21	5	2	37	22	26	61	2895	1891	1065	1061.5
58	27	9	4	24	27	36	55	2931	1946	1040	1033.5
59	26	4	2	34	20	30	56	2961	2002	1015	1012
60	26	6	2	28	17	32	47	2993	2049	991	987
61	19	4		36	21	23	57	3016	2106	967	965
62	24	3	1	40	21	27	62	3043	2168	937	935
63	22	2		41	14	24	55	3067	2223	899	898
64	10	2	1	37	17	12	55	3079	2278	856	854.5
65	20	9		40	18	29	58	3108	2336	830	825.5
66	10	2		39	18	12	57	3120	2393	784	783
67	12	2		38	20	14	58	3134	2451	741	740
68	14			29	20	14	49	3148	2500	697	697
69	13			38	14	13	52	3161	2552	661	661
70	12			36	10	12	46	3173	2598	621	621
71	19	4		48	14	23	62	3196	2660	598	596
72	10			40	15	10	55	3206	2715	546	546
73	9			50	11	9	61	3215	2776	500	500
74	7	1		49	10	8	59	3223	2835	447	446.5
75	9			44	9	9	53	3232	2888	397	397
76	11			38	9	11	47	3243	2935	355	355
77	4	1		46	13	5	59	3248	2994	313	312.5
78	2		1	47	5	2	53	3250	3047	256	255.5
79	3			26	9	3	35	3253	3082	206	206
80	1			34	9	1	43	3254	3125	172	172
81	1			19	4	1	23	3255	3148	130	130
82	2			18	6	2	24	3257	3172	109	109
83	1			16	1	1	17	3258	3189	86	86
84	2			16	2	2	18	3260	3207	71	71
85	1			11	2	1	13	3261	3220	54	54
86	1			9	1	1	10	3262	3230	42	42
87	1			7		1	7	3263	3237	33	33
88				8	1		9	3263	3246	26	26
89				3	1		4	3263	3250	17	17
90				5			5	3263	3255	13	13
91				2			2	3263	3257	8	8
92								3263	3257	6	6
93				2	1		3	3263	3260	6	6
94				1			1	3263	3261	3	3
95				1			1	3263	3262	2	2
96				1			1	3263	3263	1	1

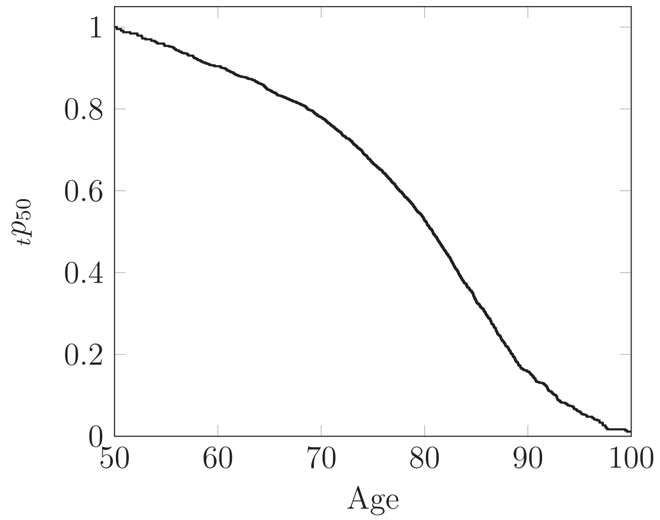
Table 4: Summary of Tables 2 and 3 to match the tables in Karup [1893, pp.2-3]. The "Persons at risk" column is an approximation of the initial exposed-to-risk from summing R_x and the deaths come from summing T_x. Note that the figure for persons at risk aged 40–49 differs by 1 from Karup's original published table. The "Mortality rate" is an annual q-type mortality rate, rather than the mortality hazard.

Age band	Persons at risk	Deaths	Mortality rate
15-29	4,420.0	29	0.66%
30-39	11,486.5	73	0.64%
40-49	12,208.5	143	1.17%
50-59	11,136.0	252	2.26%
60-69	8,346.0	366	4.39%
70-79	4,235.5	424	10.01%
80-89	740.0	141	19.05%
90-96	39.0	12	30.77%

B Appendix B

We assume that we have n records of individual lifetimes. For each life i we have the age at entry to observation, x_i, the time life i was observed, t_i, and an indicator d_i taking the value 1 if life i is dead at age $x_i + t_i$ and zero otherwise. This information is enough to calculate the product-limit estimate of the survival curve in equation (3), as shown in Figure 5.

Figure 5: Plot of product-limit estimate for the survival curve of males in a Scottish public-sector pension scheme. Experience covers ages 50 and over in the period 2000–2009, with 1,791 deaths among 7,474 lives.



Consider the fitting of a survival model using these individual data. The log-likelihood, ℓ, is:

$$\ell = -\sum_{i=1}^{n} H_{x_i}(t_i) + \sum_{i=1}^{n} d_i \log \mu_{x_i+t_i} \tag{24}$$

If we use the parameterisation of equation (9), one question is how to obtain good initial starting values for optimising ℓ in equation (24)? Following the insight of Table 1 we can use the product-limit estimates of ${}_tp_x$ by adapting the equation systems in (11) and (12). For a general outset age, x, we have:

$$a = \log {}_tp_x = -H_x(t) \quad (25)$$
$$b = \log {}_tp_{x+t} = -H_{x+t}(t) \quad (26)$$
$$c = \log {}_tp_{x+2t} = -H_{x+2t}(t) \quad (27)$$

Assuming $a \neq b \neq c$ and $a + c \neq 2b$, initial estimates for the parametric Makeham survival model are given by:

$$\begin{aligned} \beta_0 &= \frac{1}{t}\log\left(\frac{b-c}{a-b}\right) \\ \alpha_0 &= \log\left(\frac{\beta(a-b)^3}{(a+c-2b)^2}\right) - \beta x \\ \epsilon_0 &= \log\left(\frac{b^2-ac}{t(a+c-2b)}\right) \end{aligned} \quad (28)$$

We pick an outset age, x, where there are observed deaths and a value of t that spans as much of the age range above x as possible without reaching the highest ages where data are sparse and the product-limit estimate becomes volatile. For the pension scheme in Figure 5 we pick $x = 50$ and $t = 15$ years to span the age range 50–95. We find the closest-available product-limit survival probabilities as follows:

Table 5: Selected product-limit estimates from Figure 5.

Age $50 + t$	${}_tp_{50}$
64.924042	0.846281240
65.017131	0.845928328
⋮	⋮
79.996233	0.528138031
80.004447	0.527716195
⋮	⋮
94.824749	0.061170831
95.019141	0.059748254

The product-limit estimate is constant between jump ages, so the estimates of the survival probabilities are:

$$\begin{aligned} {}_{15}p_{50} &= 0.846281 \\ {}_{30}p_{50} &= 0.528138 \\ {}_{45}p_{50} &= 0.061171 \end{aligned} \quad (29)$$

and we estimate our trio of t-year survival probabilities as follows:

$${}_{15}p_{50} = {}_{15}p_{50} = 0.846281 \quad (30)$$
$${}_{15}p_{65} = \frac{{}_{30}p_{50}}{{}_{15}p_{50}} = 0.624069 \quad (31)$$
$${}_{15}p_{80} = \frac{{}_{45}p_{50}}{{}_{30}p_{50}} = 0.115824 \quad (32)$$
$$(33)$$

We therefore calculate a, b and c to be:

$$a = \log {}_{15}p_{50} = -0.166904 \quad (34)$$
$$b = \log {}_{15}p_{65} = -0.471494 \quad (35)$$
$$c = \log {}_{15}p_{80} = -2.155687 \quad (36)$$

Putting the estimates in (34)-(36) into equation system (28) we get initial values for the three parameters. These are shown in Table 6 along with the final maximum-likelihood estimates from using equation (24). Table 6 shows that using the product-limit estimate provides good initial values for fitting a survival model based on individual lifetimes.

Table 6: Comparison of (a) initial values from equation system (28), and (b) maximum-likelihood estimates using equation (24).

Methodology	$\hat{\beta}$	$\hat{\alpha}$	$\hat{\epsilon}$
(a) Initial values using equation system (28)	0.114005	-12.0817	-5.01408
(b) Maximum likelihood estimates	0.110625	-11.6892	-5.43406
Absolute relative difference (%)	3.1%	3.4%	7.7%

For the Gompertz model in equation (6) we can adapt Gompertz's own system for a general outset age, x:

$$a = \log {}_tp_x = -H_x(t) \quad (37)$$
$$b = \log {}_tp_{x+t} = -H_{x+t}(t) \quad (38)$$

where we would use values of x and t where the Gompertz assumption holds, e.g. $x = 60$ and a value of $t = 20$. Assuming $a \neq b$, the equivalent Gompertz initial values are then:

$$\begin{aligned} \beta_0 &= \frac{1}{t}\log\left(\frac{b}{a}\right) \\ \alpha_0 &= \log\left(\frac{\beta_0 a^2}{b-a}\right) - \beta_0 x \end{aligned} \quad (39)$$

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24a Ainslie Place, Edinburgh, EH3 6AJ
Telephone 0131 315 4470
Email info@longevitas.co.uk

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