A technical note on the Gompertz model of mortality with Gamma frailty

We suppose that an individual in a population of lives is subject to a force of mortality $h(x|z) = z\mu_x$ where z is the frailty of the life and x is the age of the life. Thus the frailty acts multiplicatively on μ_x; this is the model of Vaupel, Manton and Stallard (1979). Across the population the frailty is assumed to have a gamma distribution with pdf

$$f_Z(z) = \frac{k^k}{\Gamma(k)} z^{k-1} e^{-kz} \tag{1}$$

which is a gamma distribution with mean 1 and variance $1/k$. Thus, in a sense, a life age x with frailty $z = 1$ is a reference life with force of mortality μ_x. If $z < 1$ then the life has a better mortality relative to the reference life, whereas if $z > 1$ then the life has a poorer mortality. The purpose of this note is to derive the marginal force of mortality of a randomly chosen member of the population.

Lemma 1 The marginal pdf of the age of death is

$$f_X(x) = \mu_x \left(\frac{k}{k + H(x|1)} \right)^{k+1} \tag{2}$$

where $H(x|1)$ is the cumulative hazard of a life with frailty $z = 1$.

Proof The cumulative hazard $H(x|z)$ for a life age x with frailty z, ie, subject to a force of mortality $z\mu_x$, is

$$H(x|z) = \int_0^x h(t|z)\,dt = \int_0^x z\mu_t\,dt = zH(x|1) \tag{3}$$

where $H(x|1)$ is the cumulative hazard of a life with frailty 1. Thus the survivor function is given by

$$S(x|z) = \exp(-zH(x|1)) \tag{4}$$

and the conditional pdf of the age of death is

$$f_{X|Z=z}(x|z) = h(x|z)S(x|z) = z\mu_x \exp(-zH(x|1)). \tag{5}$$

Integrating over the distribution of Z we find

$$\begin{aligned}
f_X(x) &= \int_0^\infty g(z) f(x|z)\,dz && (6)\\
&= \int_0^\infty \frac{k^k}{\Gamma(k)} z^{k-1} e^{-kz} \times z\mu_x \exp(-zH(x|1))\,dz && (7)\\
&= \frac{k^k}{\Gamma(k)} \mu_x \int_0^\infty z^k \exp(-\phi z)\,dz, \quad \phi = k + H(x|1), && (8)\\
&= \frac{k^k}{\Gamma(k)} \mu_x \times \frac{\Gamma(k+1)}{\phi^{k+1}} && (9)\\
&= \mu_x \left(\frac{k}{k + H(x|1)} \right)^{k+1}. && (10)
\end{aligned}$$

We now specialize to the case when $\mu_x = \exp(\alpha + \beta x)$, ie, the reference force of mortality is Gompertz.

Lemma 2 Let $h(x|z) = z\mu_x$, where $\mu_x = \exp(\alpha + \beta x)$ and the frailty z follows a gamma distribution with mean 1 and variance $1/k$. Then

(a) The marginal pdf of the age of death is

$$f(x) = \frac{e^{\alpha+\beta x}}{\left(1 + \dfrac{e^{\alpha+\beta x} - e^{\alpha}}{\beta k}\right)^{k+1}}. \tag{11}$$

(b) The marginal survivor function is

$$S(x) = \frac{1}{\left(1 + \dfrac{e^{\alpha+\beta x} - e^{\alpha}}{\beta k}\right)^{k}}. \tag{12}$$

(c) The marginal force of mortality is

$$h(x) = \frac{e^{\alpha+\beta x}}{1 + \dfrac{e^{\alpha+\beta x} - e^{\alpha}}{\beta k}}. \tag{13}$$

Proof To prove (a) we apply lemma 1 with

$$H(x|1) = \int_0^x e^{\alpha+\beta t}\, dt = \left[\frac{1}{\beta} e^{\alpha+\beta t}\right]_0^x = \frac{1}{\beta}\left(e^{\alpha+\beta x} - e^{\alpha}\right) \tag{14}$$

and so

$$\begin{aligned} f(x) &= \left(\frac{k}{k + \dfrac{e^{\alpha+\beta x} - e^{\alpha}}{\beta}}\right)^{k+1} \exp(\alpha + \beta x) && (15) \\ &= \frac{e^{\alpha+\beta x}}{\left(1 + \dfrac{e^{\alpha+\beta x} - e^{\alpha}}{\beta k}\right)^{k+1}} && (16) \end{aligned}$$

which is (a).

To derive (b) we have

$$
\begin{aligned}
S(x) &= \int_x^\infty f_X(u)\,du &(17)\\
&= \int_x^\infty \frac{e^{\alpha+\beta u}}{\left(1+\dfrac{e^{\alpha+\beta u}-e^{\alpha}}{\beta k}\right)^{k+1}}\,du &(18)\\
&= \int_{v_0}^\infty \frac{1}{(1+v)^{k+1}} k\,dv, \quad v_0 = \frac{e^{\alpha+\beta x}-e^{\alpha}}{\beta k}, &(19)\\
&= \left[-(1+v)^{-k}\right]_{v_0}^{\infty} \text{ and result (b) follows.} &(20)
\end{aligned}
$$

Result (c) follows immediately by applying results (a) and (b) to $h(x) = f(x)/S(x)$.

Lemma 3 The following expressions for the force of mortality are equivalent:

(a)

$$h(x) = \frac{e^{\alpha+\beta x}}{1+\dfrac{e^{\alpha+\beta x}-e^{\alpha}}{\beta k}}, \tag{21}$$

(b)

$$h(x) = \frac{Be^{\beta x}}{1+De^{\beta x}}, \tag{22}$$

(c)

$$h(x) = \frac{e^{\alpha'+\beta x}}{1+e^{\alpha'+\rho+\beta x}}. \tag{23}$$

Proof In (b) set

$$B = \frac{\beta k}{\beta k e^{-\alpha}-1}; \quad D = \frac{1}{\beta k e^{-\alpha}-1} \tag{24}$$

and (b) reduces to (a).

In (c) set

$$\rho = -\log(k\beta); \quad \alpha' = \log\left(\frac{\beta k}{\beta k e^{-\alpha}-1}\right) \tag{25}$$

and (c) reduces to (a).